Rewriting modulo traced comonoid structure

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Why are we here?

We want to reason equationally with processes
with notions of feedback, copying and discarding

(e.g. digital circuits)

What should the syntax for these processes be?

How do we reason with this syntax?

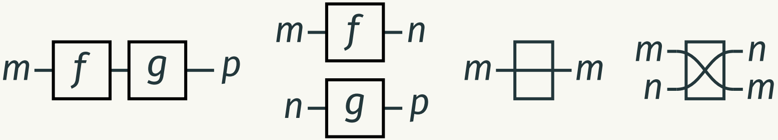
What is the best way to rewrite with this syntax?

Is this syntax suitable for automating rewrites?

The big picture

We have specialised previous work on hypergraph string diagram rewriting to settings with a traced comonoid structure.

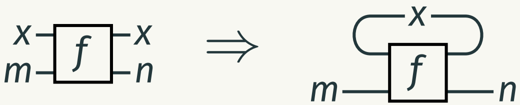
The graphical syntax of string diagrams



(symmetric monoidal category)

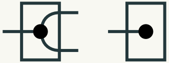
Why are we here?

We want to have feedback.



(traced structure)

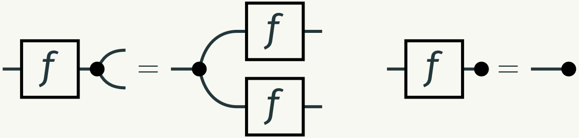
We want to fork and stub.



(commutative comonoid structure)

Why are we here?

We want to copy and discard.



(Cartesian structure)

We want to reason graphically.

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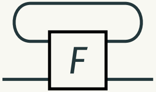

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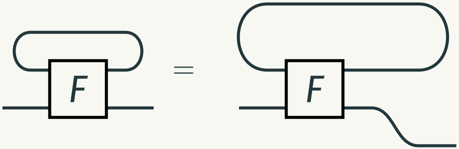

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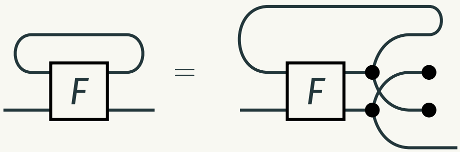

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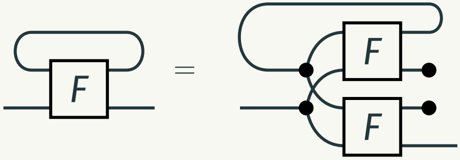

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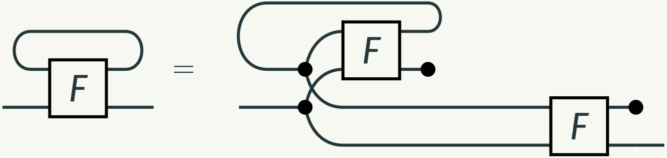

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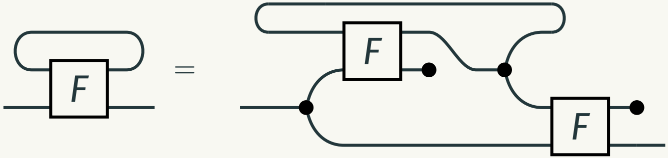

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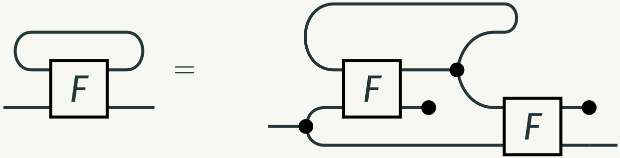

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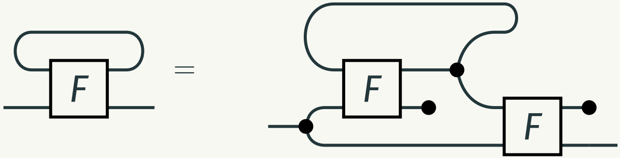


(unfolding, fixpoint equation)

We want to do this reasoning computationally.

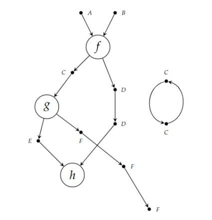
This is hard for terms, even with string diagrams.

(lots of shuffling around and bookkeeping required with the comonoid)

But computers like graphs...

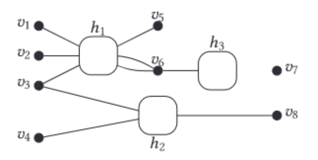
What came before

String graphs



Dixon, Kissinger

Hypergraphs

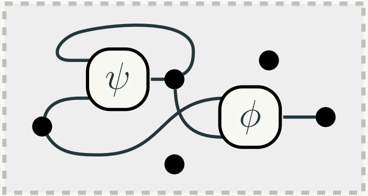


Bonchi, Gadduchi, Kissinger, Sobocinski, Zanasi

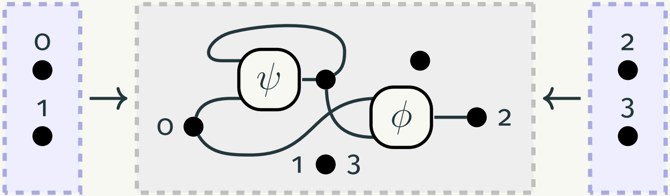




The hyper kind of graph



The hyper kind of (interfaced) graph



Goal

string diagrams as cospans of hypergraphs

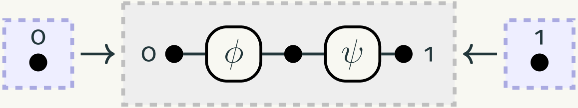
But which hypergraphs?

Keeping it single

Which cospans correspond to symmetric monoidal terms?

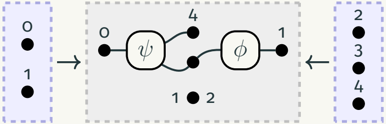
Monogamous acyclic hypergraphs

One connection on the left, one on the right



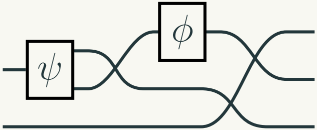
Getting the correspondence

monogamous acyclic hypergraphs



$\leftrightarrow$

symmetric monoidal term

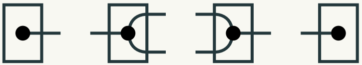


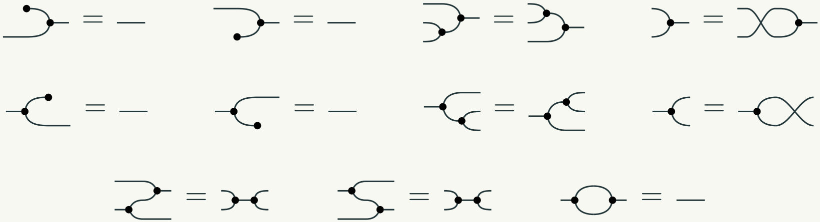
Monogamous acyclic hypergraphs are too restrictive.

Feeling special

Which terms correspond to arbitrary cospans of hypergraphs?

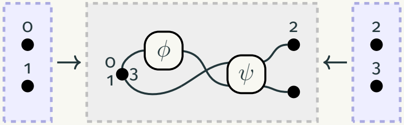
Terms with a special commutative Frobenius structure





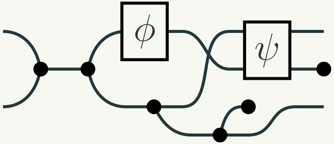
Another correspondence

isomorphism class of hypergraphs



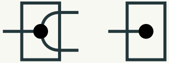
$\leftrightarrow$

Frobenius term modulo equations

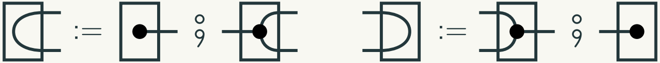


Arbitrary hypergraphs are not restrictive enough.

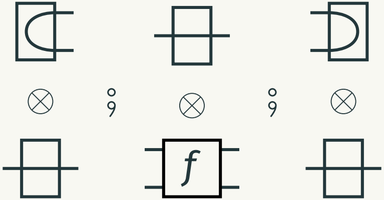
Frobenius to traced comonoid

Traced comonoid is ‘half’ Frobenius... 

Any category with Frobenius is self-dual compact closed...

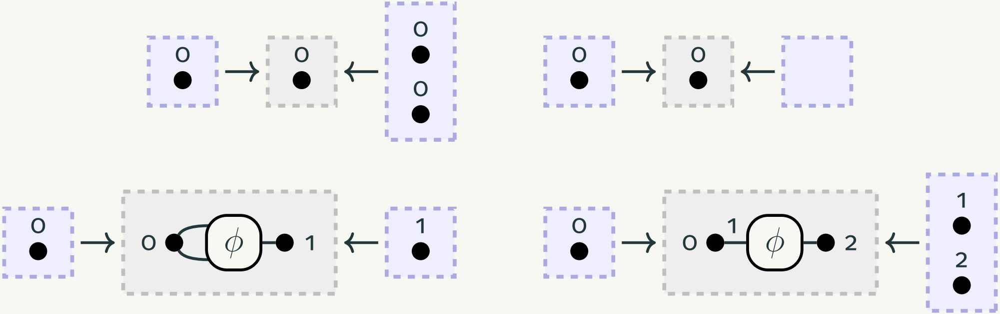


Trace can be built from compact closed structure...



Partial left-monogamous hypergraphs

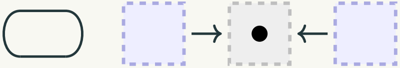
One connection on the left, many on the right



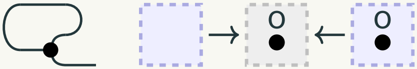
Hold on a moment

Special cases...

Trace of the identity

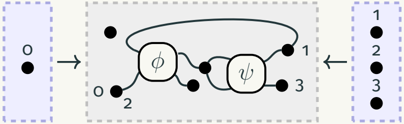


Trace of the fork



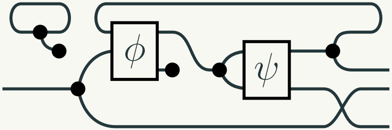
One more correspondence

partial left-monogamous hypergraphs



$\leftrightarrow$

traced comonoid term



Now what?

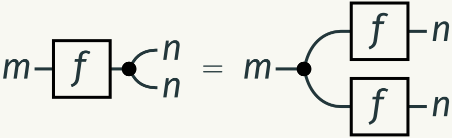
We can interpret terms as graphs

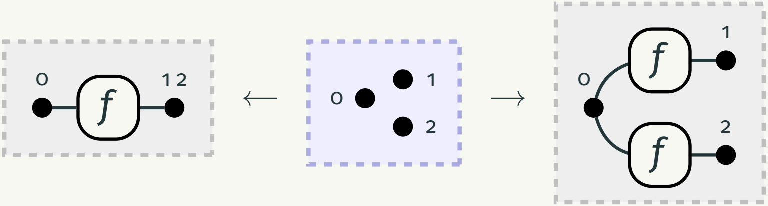
Now to reason with them!

Applying equations $\leftrightarrow$ Graph rewriting

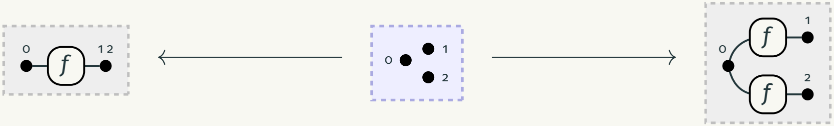
Double pushout (DPO) rewriting

One rule for them





Do the double pushout



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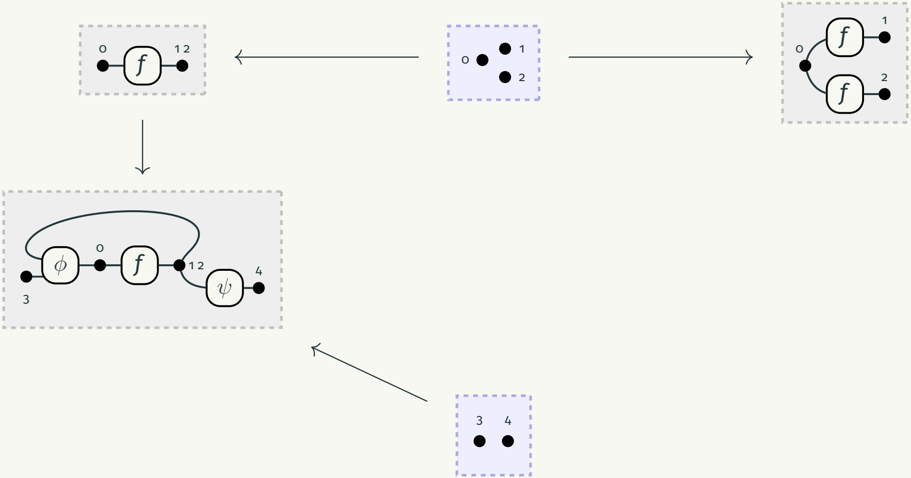
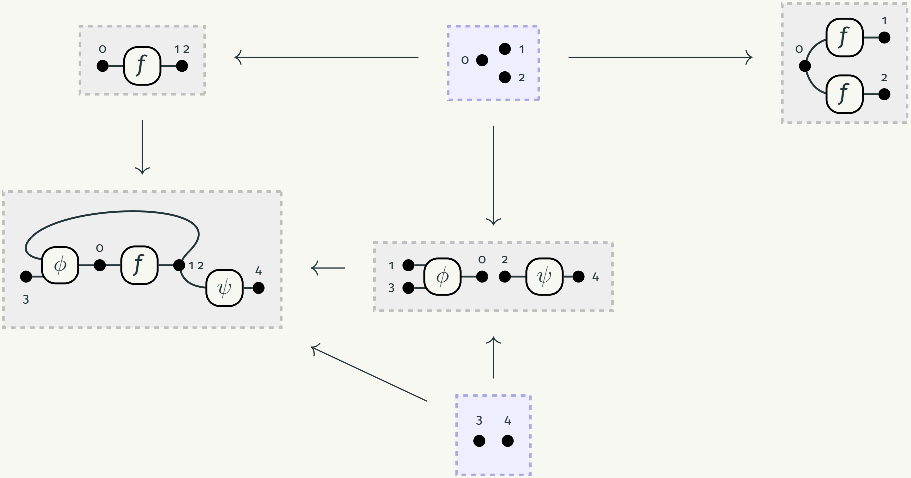


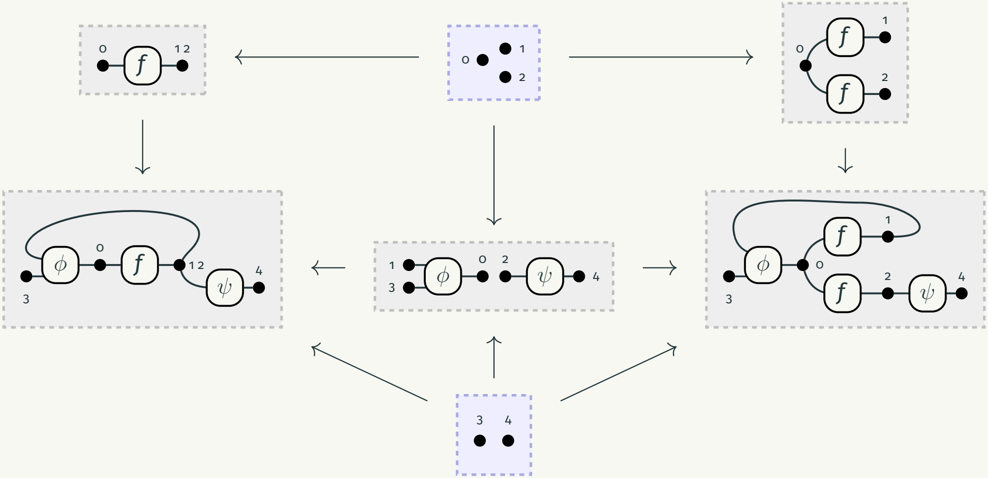
Matching

Do the double pushout



Pushout complement

Do the double pushout



Pushout!

Which pushout complements are valid rewrites?

Give me complements

Which pushout complements are valid rewrites?

Symmetric monoidal setting? Exactly one pushout complement valid

Give me complements

Which pushout complements are valid rewrites?

Symmetric monoidal setting? Exactly one pushout complement valid

Frobenius setting? All pushout complements valid

Give me complements

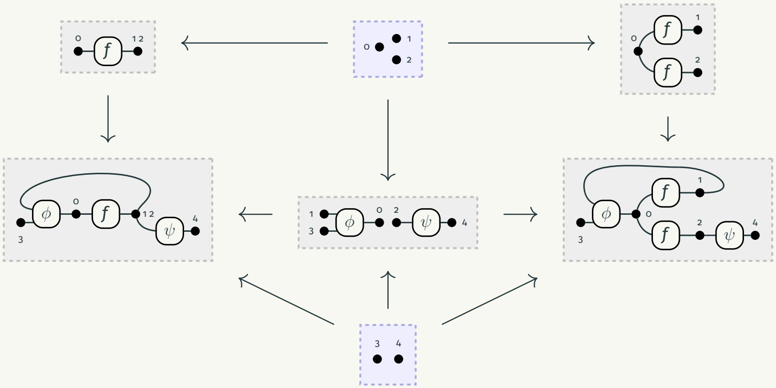
Which pushout complements are valid rewrites?

Symmetric monoidal setting? Exactly one pushout complement valid

Traced comonoid setting? Some pushout complements valid

Frobenius setting? All pushout complements valid

I need some validation



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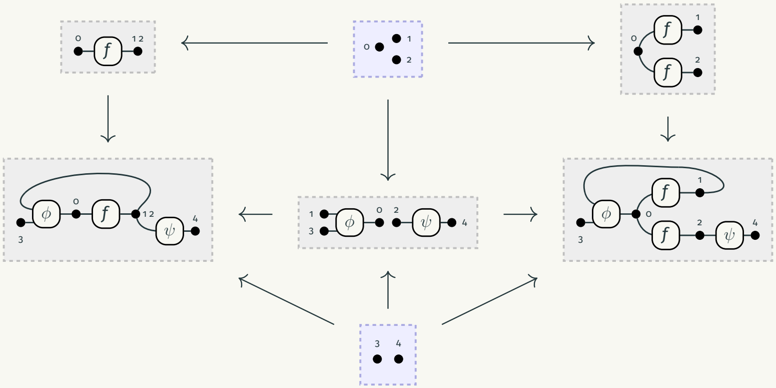
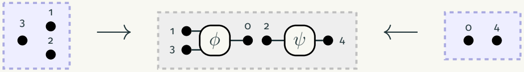


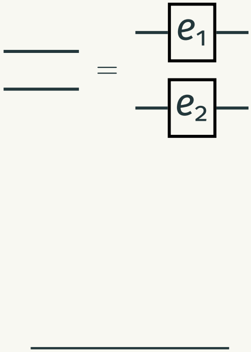
This cospan is partial left-monogamous:

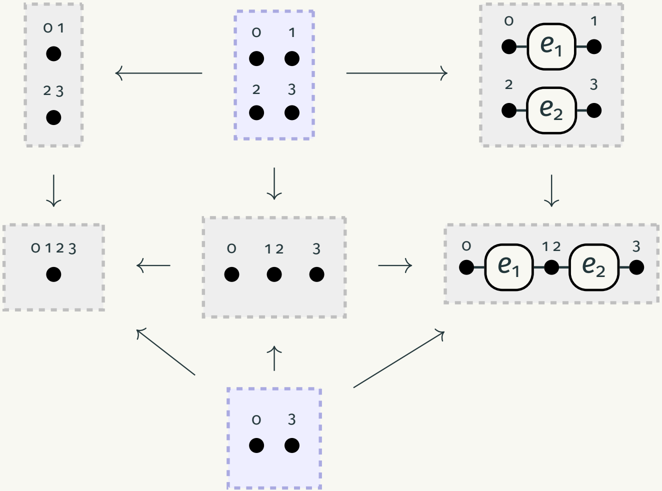
Inputs of term
Outputs of rule



Inputs of rule
Outputs of term

It’s two, actually





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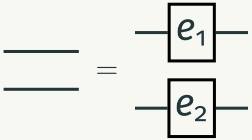

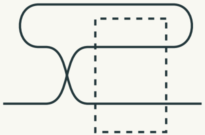

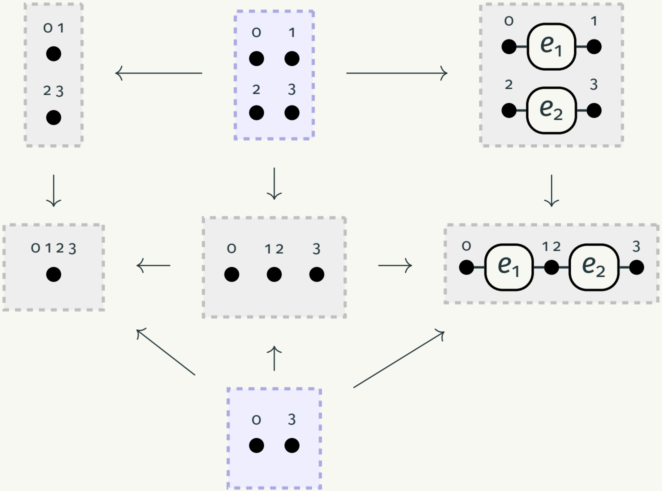

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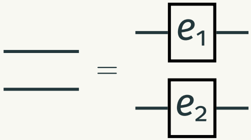

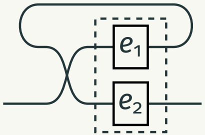

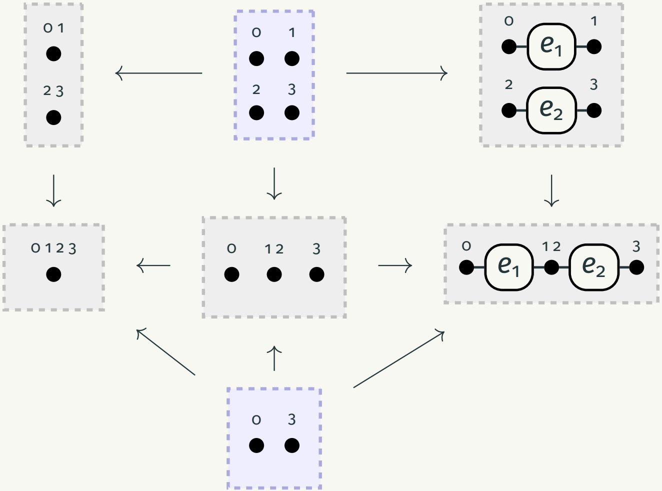

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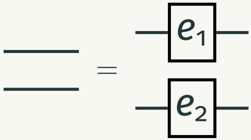

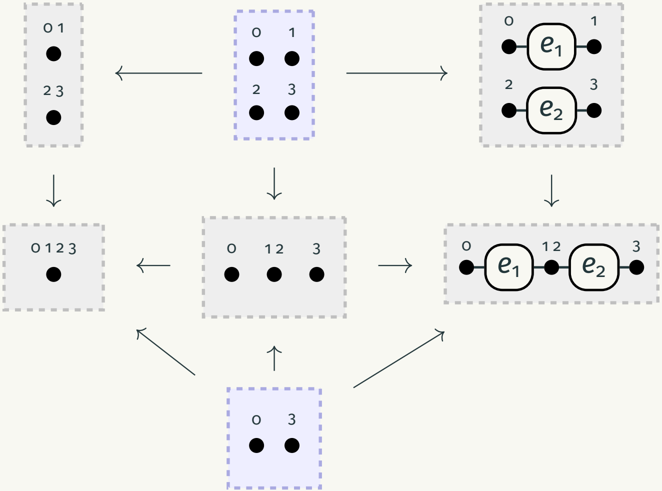

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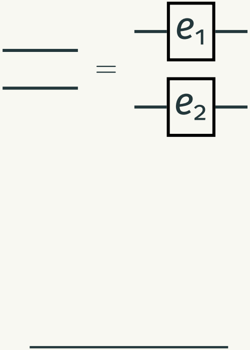

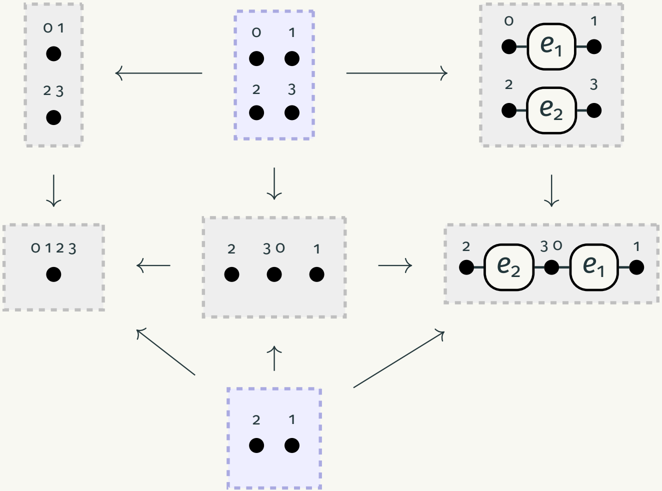

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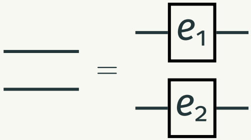

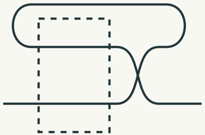

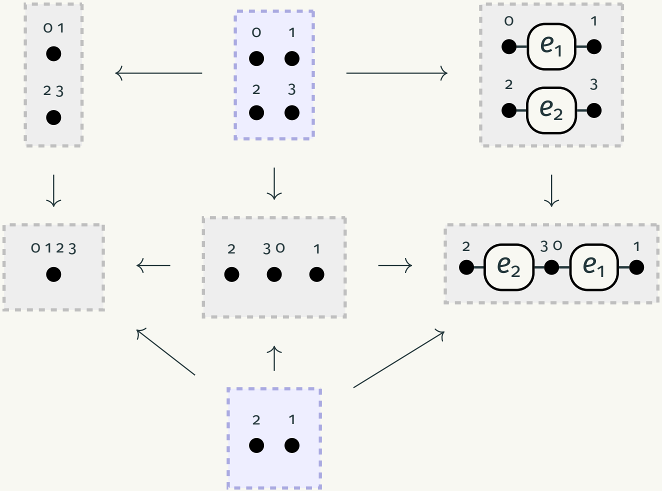

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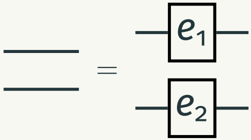

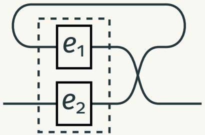

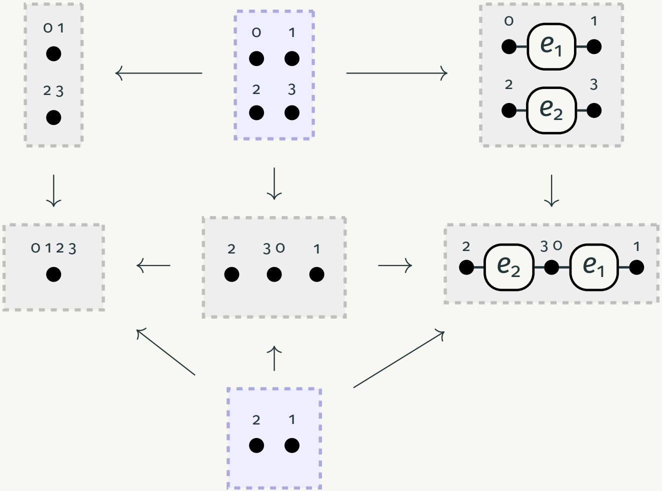

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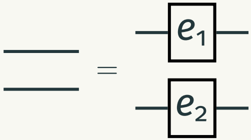
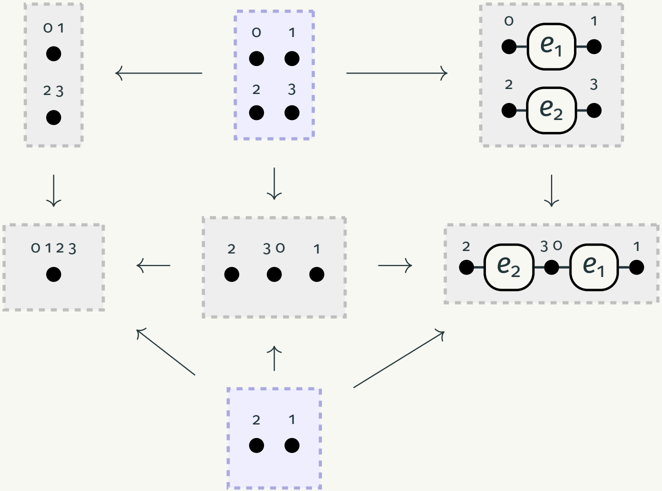
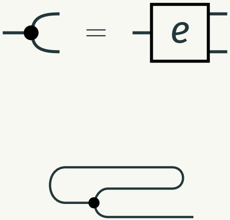
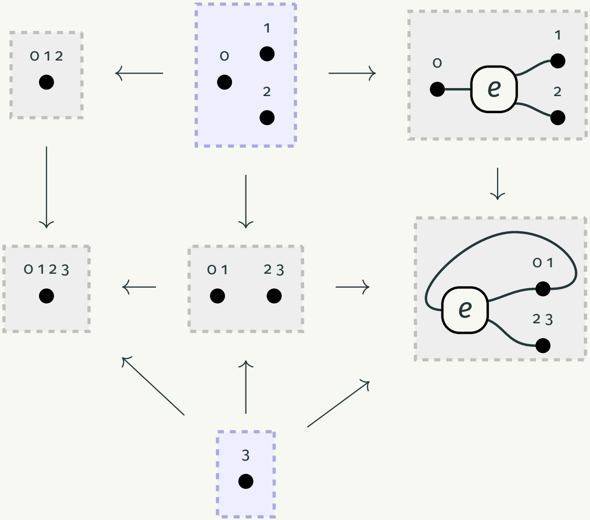




It’s two, actually (comonoid style)





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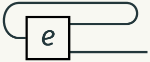

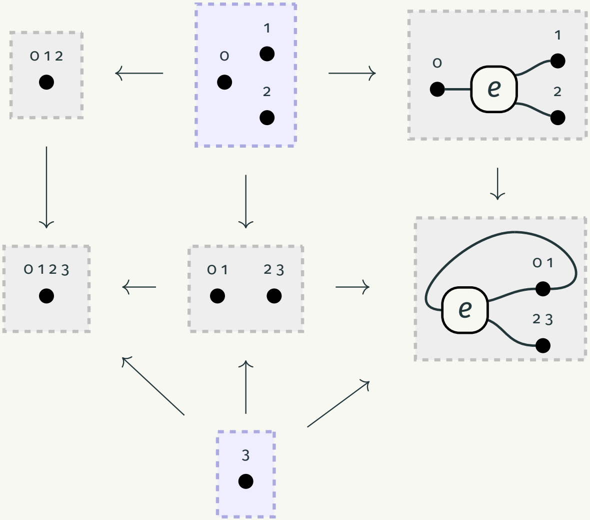

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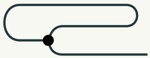

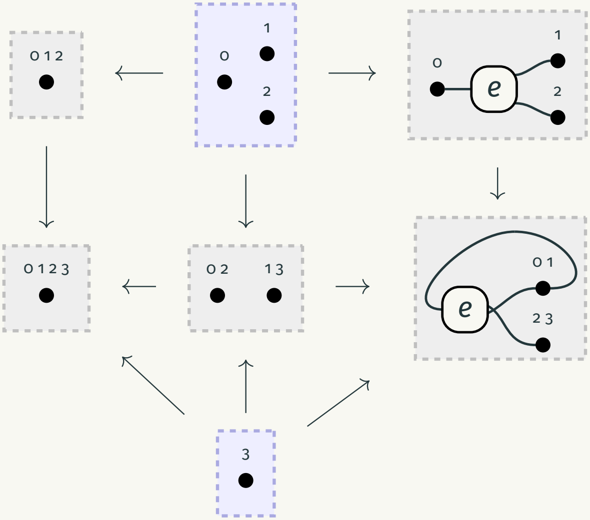

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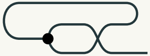
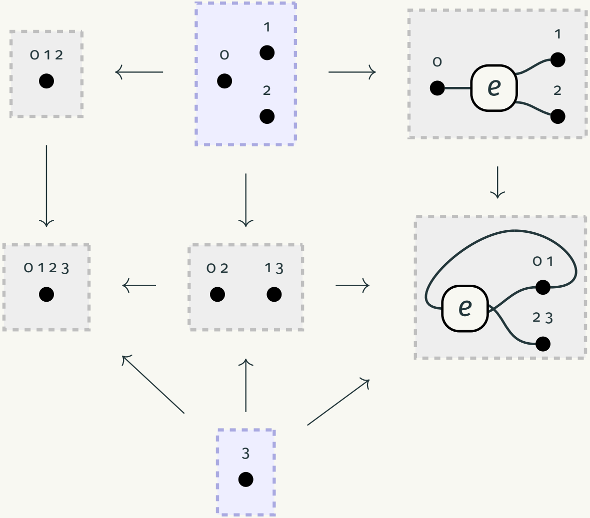
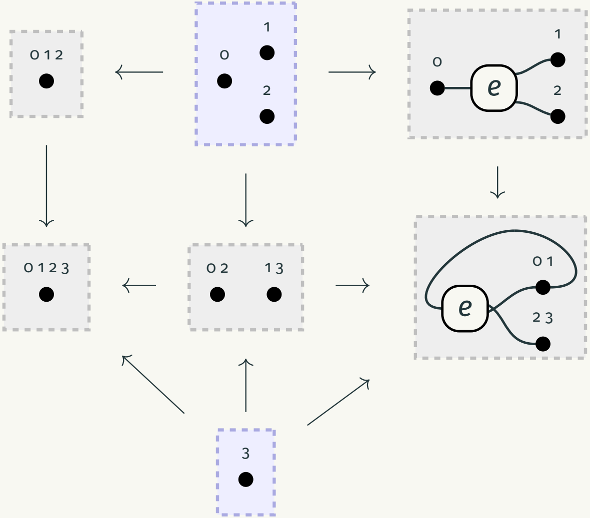
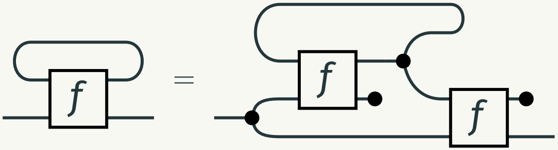




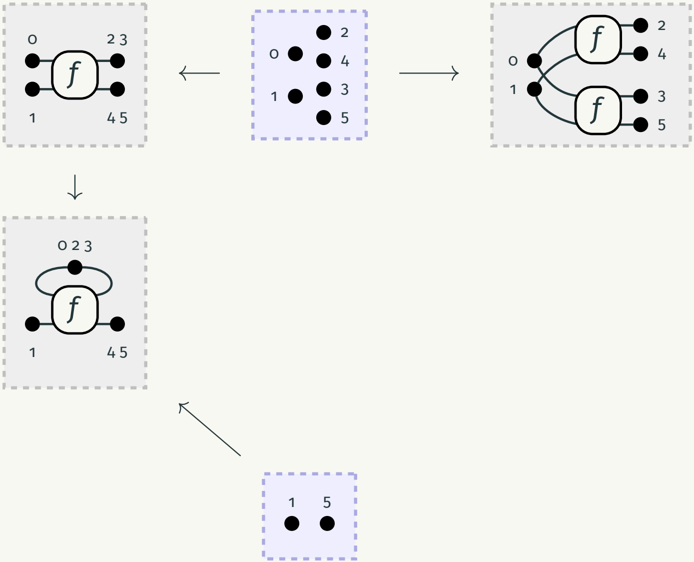
It's two, actually (comonoid style)



Remember me?



Unfolding again



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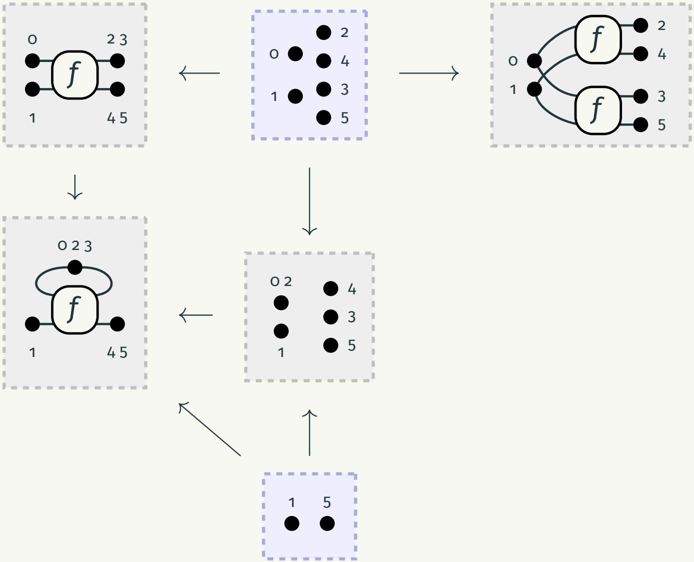

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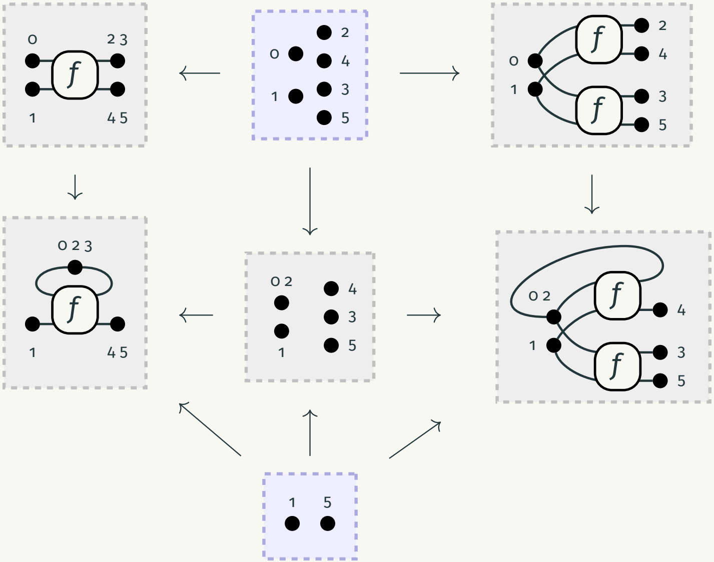

Two contributions

Characterised partial left-monogamous cospans of hypergraphs as a suitable hypergraph interpretation of traced comonoid terms

Characterised the correct notion of pushout complement for traced comonoid terms