

A Fully Compositional Theory of Sequential Digital Circuits

Denotational, Operational, and Algebraic Semantics

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Abstract

Digital circuits, despite having been studied for nearly a century and used at scale for about half that time, have until recently evaded a fully compositional theoretical understanding, in which arbitrary circuits may be freely composed together without consulting their internals. Recent work remedied this theoretical shortcoming by showing how digital circuits can be presented compositionally as morphisms in a freely generated symmetric traced category. However, this was done informally; in this paper we refine and expand the previous work in several ways, culminating in the presentation of three sound and complete semantics for digital circuits: denotational, operational and algebraic. For the denotational semantics, we establish a correspondence between stream functions with certain properties and circuits constructed syntactically. For the operational semantics, we present the reductions required to model how a circuit processes a value, including the addition of a new reduction for eliminating non-delay-guarded feedback; this leads to an adequate notion of observational equivalence for digital circuits. Finally, we define a new family of equations for translating circuits into bisimilar circuits of a ‘normal form’, leading to a complete algebraic semantics for sequential circuits.

1 Introduction

Bothe was awarded the 1954 Nobel Prize in physics for creating the electronic AND gate in 1924. In the ensuing decades, exponential improvements in digital technology have led to the development of the defining technologies of the modern world. It may therefore seem improbable that there are theoretical gaps remaining in our mathematical and logical understanding of digital circuits.

To be more precise, by ‘digital circuits’ we primarily understand *electronic* circuits: deterministic circuits with clear notions of input and output and which work on discrete signals. A classic example is that of logical gates and basic memory elements of known and fixed delays, but there are alternatives such as CMOS transistors operating in saturation mode. What we do not attempt to handle are circuits operating on continuous signals (such as amplifiers) or in continuous time (such as asynchronous circuits), nor *electrical* circuits of resistors and capacitors, which are quite different [BS22].

Our goal is to devise a *fully compositional* model of digital circuits. By ‘fully compositional’ we mean that a larger circuit can be built from smaller circuits and interconnecting wires without paying heed to the internal structure of these smaller circuits. Of course, composition comes naturally to digital circuits and is widely used informally [Gor82]. Unfortunately one runs into an obstacle when trying to formalise this notion mathematically: electrical connections can be created that inadvertently connect the output of some elementary gate back to its input such that no memory elements are encountered along the path. Such a path, called a ‘combinational feedback loop’ (or ‘cycle’), is not handled by established mathematical theories of digital circuits, so conventional digital design and engineering

reject such circuits. To enforce this restriction, we need to always look inside circuits as we compose them, ensuring that no illegal combinational feedback loops are created, or resort to some ‘safe’ subset of circuits [Chr+21]. This represents a failure of compositionality: what we want to do is to compose *any* circuits constructed from a fixed set of components.

On general principle, we have reason to expect that a compositional theory of digital circuits may lead to more streamlined methods of analysis and verification, which, in time, may also lead to new applications. *Combinational* circuits, which model functions, have an obvious compositional syntax [Laf03], but *sequential* circuits, which contain delay and feedback, are more subtle. The first forays towards a fully compositional syntactic and categorical account of circuits have been made recently [GJ16; GJL17], but they do not paint a fully formal and coherent picture. This paper develops the informal presentation into a mathematically rigorous framework.

Our first contribution is to give, for the first time, a sound and complete denotational semantics to digital circuits in the domain of causal and monotone stream functions. The completeness result depends on a novel albeit straightforward lifting of Mealy machines [Mea55] to act on alphabets with a lattice structure, utilising a handy coalgebraic perspective [Rut06]. Using Mealy machines to give a semantic interpretation to digital circuits is an established methodology [KJ09], and here they act as a ‘bridge’ between the syntactic and the semantic domain, showing how existing circuit methodologies are compatible with our rigorous mathematical framework.

The second contribution of the paper is to generalise and systematise previous efforts [GJL17] to formulate a graph-rewriting-based operational semantics for digital circuits. The novelty is a new reduction rule for eliminating non-delay-guarded feedback using a version of the Kleene fixpoint theorem, thus solving the problem of productivity that previous operational semantics only solve partially. The denotational and operational semantics together achieve the long-standing goal of creating a semantic theory of digital circuits using the same methodology as programming languages.

The methodological ‘glue’ that binds together the two approaches is a new sound and complete algebraic semantics, the third and final contribution of the paper. This approach replaces the previous ad-hoc way of introducing equations for digital circuits based on raw intuitions with a systematic approach guided by the denotational semantics. The key technical result of this method is deriving pseudo-normal forms of digital circuits.

Although the motivation of this work is foundational, there are some early hints of possible exciting applications, such as for partial evaluation and blackboxing. These are not yet industrial-strength applications, but the simplicity and power of the framework must hold a certain degree of appeal and promise.

2 Syntax

Let us first recall the syntax of digital circuits [GJ16].

Definition 1 (Circuit signature). *A circuit signature Σ is a tuple $(\mathbf{V}, \bullet, \mathcal{P}, \text{dom}, \text{cod})$ where \mathbf{V} is a finite set of values with a distinguished element \bullet , \mathcal{P} is a (usually finite) set of primitives, and $\text{dom}, \text{cod} : \mathcal{P} \rightarrow \mathbb{N}$ are arity and coarity functions respectively.*

We use an arbitrary set of values rather than restricting to the traditional ‘true’ and ‘false’ so the framework can model circuits at multiple layers of abstraction. For example, the values could contain ‘weak’ and ‘strong’ values, as in metal-oxide-semiconductor field-effect transistors (MOSFET). Alternatively, one could work at a higher level of abstraction and set the values to be natural numbers, for working with arithmetic circuits. All we require is a distinct element \bullet for a *disconnected wire*: a lack of information.

A particularly important signature is that of gate-level circuits, a common level of abstraction for digital circuits.

Example 2 (Gate-level circuits). *The gate-level circuit signature is $\Sigma_{\mathbf{B}} = (\mathbf{V}_{\mathbf{B}}, \perp, \mathcal{P}_{\mathbf{B}}, \text{dom}_{\mathbf{B}}, \text{cod}_{\mathbf{B}})$, where $\mathbf{V}_{\mathbf{B}} := \{\perp, f, t, \top\}$, respectively representing *no signal*, a *false signal*, a *true signal* and *both signals*, $\mathcal{P}_{\mathbf{B}} := \{\text{AND}, \text{OR}, \text{NOT}\}$, $\text{dom}_{\mathbf{B}} := \text{AND} \mapsto 2, \text{OR} \mapsto 2, \text{NOT} \mapsto 1$, and $\text{cod}_{\mathbf{B}} := - \mapsto 1$.*

A circuit signature freely generates symmetric monoidal categories (SMCs) of digital circuits. These are PROPS (categories of *PRO*ducts and *PER*mutations) [Mac65], SMCs with natural numbers as objects and addition as tensor product on objects.

Instead of term strings, we employ the two dimensional syntax of *string diagrams* [JS91; JSV96; Sel11]. We draw an arbitrary generator $\varphi : m \rightarrow n$ as a box $m \boxed{\varphi} n$ with m input wires and n output wires. To avoid clutter and to enable reasoning about diagrams with arbitrary inputs and outputs, wires may be

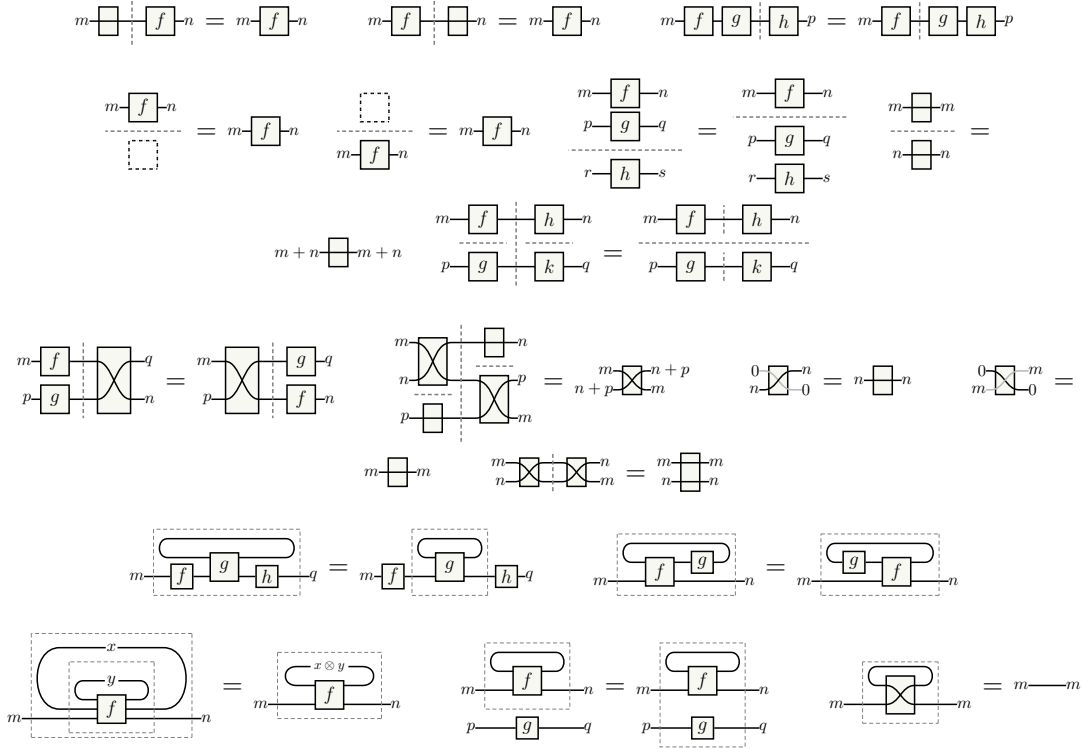


Figure 1: The equations of traced PROPS, represented in string diagram notation

collapsed into one wire and labelled appropriately $m\text{-}\boxed{f}\text{-}n$. Here this is purely notational, but this idea has been formalised syntactically elsewhere [WGZ23].

Composite morphisms, or ‘terms’, are drawn as wider boxes $m\text{-}\boxed{f}\text{-}n$; composition as horizontal juxtaposition $m\text{-}\boxed{f}\text{-}\boxed{g}\text{-}n$ and tensor product as vertical juxtaposition $m\text{-}\boxed{f}\text{-}n$, $p\text{-}\boxed{g}\text{-}q$. One of the advantages of this notation over standard term syntax is that structural rules (identity, associativity, functoriality) are ‘absorbed’ into the diagrams, as illustrated in Fig. 1.

Circuits with no delay and feedback are known as *combinational* circuits; these circuits implement *functions*.

Definition 3 (Combinational circuits). *Given a circuit signature $\Sigma = (\mathbf{V}, \bullet, \mathcal{P}, \text{dom}, \text{cod})$, let CCirc_Σ be the symmetric strict monoidal prop generated freely over*

$$\text{dom}(\phi)\text{-}\boxed{\phi}\text{-}\text{cod}(\phi) \text{ for each } \phi \in \mathcal{P}, \boxed{\bullet}, \boxed{\leftarrow}, \boxed{\rightarrow} \text{ and } \boxed{\circ}.$$

Each symbol in the signature defines a generator in CCirc_Σ . The remaining generators are *structural* generators for manipulating wires, present regardless of the signature. In order, they are for *introducing* wires, *forking* wires, *joining* wires and *eliminating* wires.

Example 4. The CCirc_{Σ_B} gates are $\boxed{\leftarrow}$, $\boxed{\rightarrow}$, and $\boxed{\circ}$.

Arbitrary combinational circuits are drawn as boxes with *light blue* backgrounds $m\text{-}\boxed{f}\text{-}n$, but when drawing generators explicitly the coloured backgrounds will often be omitted in the interests of clarity. Since the category is freely generated, morphisms are defined by juxtaposing the generators in a given signature, the identity and the symmetry, sequentially or in parallel.

Notation 5. *It is straightforward to define arbitrary-bit versions of the structural generators using the axioms of SMCs. In diagrams, these are drawn the same way as their single-bit counterparts:*

$$\boxed{\bullet}\text{-}n \quad n\text{-}\boxed{\leftarrow}\text{-}n \quad n\text{-}\boxed{\rightarrow}\text{-}n \quad n\text{-}\boxed{\circ} \quad n\text{-}\boxed{\leftarrow}\text{-}n \quad m\text{-}\boxed{\leftarrow}\text{-}n$$

Combinational circuits have no internal state. Real-world circuits often involve *delay* and *feedback*: these are known as *sequential circuits*. To model feedback, extra structure must be added to the category of combinational circuits in the form of a *trace*.

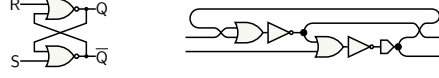


Figure 2: An SR NOR latch, and a construction in $\mathbf{SCirc}_{\Sigma_B}$.

Definition 6 ([JSV96], Sec. 2; [Has09], Sec. 3). A symmetric traced monoidal category, often abbreviated as STMC, is a SMC \mathcal{C} equipped with a family of functions $\text{Tr}_{A,B}^X(-): \mathcal{C}(X \otimes A, X \otimes B) \rightarrow \mathcal{C}(A, B)$ satisfying the axioms of STMCs listed in Fig. 1.

In string diagrams, the trace is represented by joining some of the inputs of a circuit to its outputs.

$$\text{Tr}_{m,n}^x \left(\begin{array}{c} x \\ \boxed{F} \\ -x \\ m \end{array} \right) \stackrel{\text{def}}{=} \begin{array}{c} \boxed{F} \\ \boxed{F} \\ m \end{array}$$

Definition 7 (Sequential circuits). Let \mathbf{SCirc}_{Σ} be the STMC freely generated over the generators of \mathbf{CCirc}_{Σ} , as given in Definition 3, along with new generators \boxed{v} for each $v \in \mathbf{V} \setminus \bullet$, and $\boxed{\square}$.

Morphisms in \mathbf{SCirc}_{Σ} are distinguished from those in \mathbf{CCirc}_{Σ} by a darker green colouring \boxed{F} . The additional generators introduce *state* into circuits. The smaller generators with no inputs are *instantaneous values*: the initial state of a circuit. We will use ‘value’ to refer to the generator $\boxed{\bullet}$ in addition to any sequential generators \boxed{v} .

Example 8. The values of $\mathbf{SCirc}_{\Sigma_B}$ are $\boxed{\bullet}$, \boxed{f} , \boxed{t} and \boxed{T} ; the first is combinational and the others are sequential.

Note that every circuit in \mathbf{CCirc}_{Σ} is also a circuit in \mathbf{SCirc}_{Σ} .

Remark 9. The pentagon is a delay generator. The mathematical interpretation of a delay is straightforward: it is the gap between one element of a stream and the next element. What physical aspect of a digital circuit a delay models is more flexible. The simplest interpretation is to think of a delay simply as a D flipflop in a clocked circuit, case in which the delay is one clock cycle. However, there is a more subtle interpretation in which we can think of the delay as a minimum observable duration, which can be used to model inertial delay on wires, up to some fixed precision. In the rest of the paper, unless otherwise specified, the intended physical interpretation is the former.

Example 10 (SR latch). One example of a delay being used to model wire delay is in an *SR NOR latch*, shown in Fig. 2; a NOR gate $\boxed{\text{NOR}}$ is defined as $\boxed{\text{NOR}} := \boxed{\text{NOR}} \boxed{\square}$. SR latches are used to hold state: when the S input is pulsed true the Q output will be held at true until the R input is true. SR latches work because of delays in how gates and wires transmit signals; one of the feedback loops between the two NOR gates will ‘win’. In \mathbf{SCirc}_{Σ} this is modelled by using a different number of delay generators on the wires between the top and the bottom of the latch, as shown in Fig. 2.

Notation 11. As in \mathbf{CCirc}_{Σ} , it is useful to reason with multiple sequential generators in parallel. Parallel delays will be drawn as $\boxed{\square}^n$ and, for a word $\bar{v} \in \mathbf{V}^n$, parallel values as $\boxed{\bar{v}}^n$.

3 Denotational semantics

Circuits in \mathbf{SCirc}_{Σ} are purely *syntax*: they currently have no behaviour associated with them. In this section we will present a fully compositional denotational semantics for sequential circuits based on *causal*, *monotone* and *finitely specified* stream functions. This denotational semantics is constructed *compositionally* using a functor from \mathbf{SCirc}_{Σ} to a PROP of stream functions with the desired properties; a reverse functor is then defined which maps a stream function f to a circuit in \mathbf{SCirc}_{Σ} with f as its denotation, showing that this denotational semantics is *sound and complete*.

Remark 12. In [MSB12], the semantics of digital circuits with delays cycles are presented using *timed ternary simulation*, an algorithm to compute how a sequence of circuit outputs stabilises over time given the inputs and value of the current state. Essentially, one must solve a system of equations in terms of the nodes inside a circuit to determine its behaviour. Our approach is different as we assign each circuit a concrete stream function describing its behaviour, and show that every such stream function is the behaviour of at least one circuit in \mathbf{SCirc}_{Σ} .

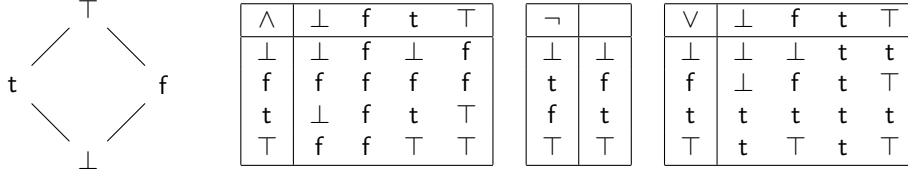


Figure 3: The lattice structure on \mathbf{V}_B , and the truth tables of Belnap logic gates [Bel77].

Recall that a function f between two posets is *monotone* if $x \leq y \Rightarrow f(x) \leq f(y)$, and a *lattice* is a poset in which each pair of elements has a least upper bound \vee (a *join*) and a greatest lower bound \wedge (a *meet*); subsequently every *finite* lattice has an *infimum* \perp and a *supremum* \top . We write v^n for the n -tuple containing only v , and call a function $f: \mathbf{V}^m \rightarrow \mathbf{V}^n$ \perp -*preserving* if $f(\perp^m) = \perp^n$.

Definition 13 (Interpretation). *An interpretation is a tuple $\mathcal{I} = (\Sigma, \sqsubseteq, \llbracket - \rrbracket)$ where $\Sigma := (\mathbf{V}, \bullet, \mathcal{P}, \text{dom}, \text{cod})$ is a circuit signature, $(\mathbf{V}, \sqsubseteq)$ is a lattice with \bullet as the infimum, and $\llbracket - \rrbracket$ maps each $p \in \mathcal{P}$ to a \perp -preserving monotone function $\mathbf{V}^{\text{dom}(p)} \rightarrow \mathbf{V}^{\text{cod}(p)}$.*

In an interpretation, the finite set of values is lifted to a lattice. The higher a value in the lattice, the more information it specifies.

Example 14. Recall the signature $\Sigma_B = (\mathbf{V}_B, \perp, \mathcal{P}_B, \text{dom}_B, \text{cod}_B,)$ from Ex. 2. The lattice $(\mathbf{V}_B, \sqsubseteq_B)$ is defined as in the diagram in the left of Fig. 3. The gates are interpreted using the Belnap tables [Bel77] in Fig. 3. Let $\llbracket - \rrbracket_B := \{\text{AND} \mapsto \wedge, \text{OR} \mapsto \vee, \text{NOT} \mapsto \neg\}$; the Belnap interpretation is defined as $\mathcal{I}_B = (\Sigma_B, \sqsubseteq_B, \llbracket - \rrbracket_B)$

Semantics are expressed formally using a *PROP morphism*, a strict symmetric (traced) monoidal functor between (traced) PROPs that is the identity on objects. When the domain of a functor is freely generated, it can be defined solely by its action on the generators.

Definition 15. Let $\mathbf{Func}_{\mathcal{I}}$ be the PROP in which the morphisms $m \rightarrow n$ are the monotone \perp -preserving functions $\mathbf{V}^m \rightarrow \mathbf{V}^n$.

Definition 16. Let $\llbracket - \rrbracket_{\mathcal{I}}^C: \mathbf{CCirc}_{\Sigma} \rightarrow \mathbf{Func}_{\mathcal{I}}$ be the PROP morphism with its action defined as

$$\begin{aligned} \llbracket \text{fork} \rrbracket_{\mathcal{I}}^C &:= \bar{x} \mapsto \llbracket p \rrbracket(\bar{x}) \\ \llbracket \text{join} \rrbracket_{\mathcal{I}}^C &:= (v) \mapsto () \\ \llbracket \text{join} \rrbracket_{\mathcal{I}}^C &:= () \mapsto (\perp) \\ \llbracket \text{fork} \rrbracket_{\mathcal{I}}^C &:= (v) \mapsto (v, v) \\ \llbracket \text{join} \rrbracket_{\mathcal{I}}^C &:= (v, w) \mapsto (v \sqcup w) \end{aligned}$$

Remark 17. One may wonder why the fork and join have different semantics, as they would be physically realised by the same wiring. This is because digital circuits have a notion of *causality*; outputs can only connect to inputs. In real life one could connect two digital devices together ignoring causality, but this might lead to undefined behaviour in the digital realm. This is modelled by the use of the join in the lattice: for example in the Belnap signature if one tries to join together t and f, the overspecified \top value will be produced.

The outputs of sequential circuits may depend on previous inputs. Their inputs are thus *streams*, infinite sequences of values. Given a set M , we denote the set of streams of M by M^ω . A stream can equivalently be viewed as a function $\mathbb{N} \rightarrow M$; consequently we write $\sigma(k)$ for the k th element of a stream $\sigma \in M^\omega$. There are two important operations used to reason with streams.

Definition 18 (Operations on streams). *The initial value is a function $\text{hd}(-): M^\omega \rightarrow M := \sigma \mapsto \sigma(0)$, producing the ‘head’ of a stream; and the stream derivative is a function $\text{tl}(-): M^\omega \rightarrow M^\omega := \sigma \mapsto (i \mapsto \sigma(i+1))$, producing its ‘tail’.*

These operations can define streams: for an element $x \in M$ and stream $\sigma \in M^\omega$, the stream $x :: \sigma$ is the unique stream with initial value x and stream derivative σ .

Sequential circuits are semantically interpreted as *stream functions*, which consume streams of inputs and produce streams of outputs. In particular, we will define the semantics of a circuit as a causal (Def. 19), monotone (Def. 22), and finitely specified (Def. 21) stream function. We begin with *causality*, which states that the output of a circuit depends only on the inputs it has seen ‘so far’.

Definition 19 (Causal stream function [Rut06]). A stream function $f: M^\omega \rightarrow N^\omega$ is causal if for all $i \in \mathbb{N}$ and all $\sigma, \tau \in M^\omega$, $\sigma(j) = \tau(j)$ for all $j \leq i$ implies $f(\sigma)(i) = f(\tau)(i)$.

Causality is a form of continuity; a stream function being causal means the i th element of the output stream depends only on inputs 0 through i . This allows the initial value and derivative operations defined for streams to be extended to *causal* stream functions.

Definition 20 (Functional stream derivative [Rut06]). Fix a causal stream function $f: M^\omega \rightarrow N^\omega$. Given $a \in M$, the initial output of f on input a is $f[a] := \text{hd}(f(a :: \sigma)) \in N$ for arbitrary $\sigma \in M^\omega$. The functional stream derivative of f on input a is the function $\partial_a f: M^\omega \rightarrow N^\omega := \sigma \mapsto \text{tl}(f(a :: \sigma))$. In the upcoming results, we may abbreviate $\partial_a f$ to f_a .

The causality of f ensures $f[a]$ does not depend on the choice of σ . $\partial_a f$ can be thought of as acting as f would ‘had it seen the input a first’. This can be extended to finite words of elements of M ; the set of such sequences is denoted M^* and the empty word as ε . Arbitrary length words are written with an underline, e.g. $\underline{v} := \text{t} :: f \in \mathbf{V}^*$; words of words of fixed width are written additionally with an overline, e.g. $\overline{v} := (\text{t}, f) :: (f, \text{t}) \in (\mathbf{V}^2)^*$.

Definition 21. For a causal stream function $f: M^\omega \rightarrow N^\omega$, we define $f_{\underline{w}}$ for $\underline{w} \in M^*$ by induction on the length of \underline{w} . If \underline{w} is the empty word, $f_{\underline{w}} = f$. Otherwise we can write $\underline{w} = \underline{u} :: a$, in which case $f_{\underline{w}} = \partial_a f_{\underline{u}}$. We say f is finitely specified if the set $\{f_{\underline{w}} : \underline{w} \in M^*\}$ is finite.

Since circuits are built from components whose interpretations are monotone functions, their interpretations as stream functions must also be monotone.

Definition 22. For a partially ordered set M and streams $\sigma, \tau \in M^\omega$, we say $\sigma \leq_{M^\omega} \tau$ if $\sigma(k) \leq_M \tau(k)$ for all $k \in \mathbb{N}$. A causal stream function $f: M^\omega \rightarrow N^\omega$ is monotone if it is monotone with respect to the above orderings on M^ω and N^ω .

We may drop the subscripts on these orders when they are obvious from context.

It is now possible to assemble a traced PROP of stream functions that correspond to sequential circuits. Given words $\overline{v} \in X^m, \overline{w} \in X^n$, we write $\overline{vw} \in X^{m+n}$ for their concatenation; abusing notation, given two streams $\sigma \in (X^m)^\omega, \tau \in (X^n)^\omega$, we also write $\sigma\tau \in (X^{m+n})^\omega$ for their pointwise concatenation. For a stream $\sigma \in (X^{m+n})^\omega$, we write $\pi_0(\sigma) \in (X^m)^\omega$ for the stream of words containing the first m characters of words in σ , and $\pi_1(\sigma) \in (X^n)^\omega$ for the stream of words containing the last n characters.

Lemma 23. Causality, monotonicity and being finitely specified is preserved by composition and tensor product.

Proof. For causality, if the i th element of two stream functions f and g only depends on the first $i + 1$ elements of the input, then so will their composition. For finitely many stream derivatives, both the composition and product of two stream functions f and g , the largest the set of stream derivatives could be is the product of stream derivatives of f and g , so this will also be finite. Finally, the composition and product of any monotone function is monotone. \square

Definition 24. Let $\mathbf{Stream}_{\mathcal{I}}$ be the PROP in which the morphisms $m \rightarrow n$ are the causal, monotone and finitely specified stream functions $f: (\mathbf{V}^m)^\omega \rightarrow (\mathbf{V}^n)^\omega$.

To model the semantics of circuits, $\mathbf{Stream}_{\mathcal{I}}$ needs a trace.

Proposition 25. Let $f: (\mathbf{V}^{x+m})^\omega \rightarrow (\mathbf{V}^{x+n})^\omega$ be a morphism in $\mathbf{Stream}_{\mathcal{I}}$. For each $\sigma \in (\mathbf{V}^m)^\omega$, let $\mu_f(\sigma)$ be the least fixed point of the endofunction $\tau \mapsto \pi_0(f(\tau\sigma))$; a trace $\text{Tr}^x(f): (\mathbf{V}^m)^\omega \rightarrow (\mathbf{V}^n)^\omega$ is defined by $(\text{Tr}^x(f))(\sigma) := \pi_1(f((\mu_f(\sigma))\sigma))$.

Proof. We must first show that $\sigma \mapsto \pi_1(f(\mu_f(\sigma)\sigma))$ is in $\mathbf{Stream}_{\mathcal{I}}$: it is causal, finitely specified, and monotone.

Since $f: (\mathbf{V}^{x+m})^\omega \rightarrow (\mathbf{V}^{x+n})^\omega$ is a morphism of $\mathbf{Stream}_{\mathcal{I}}$, it has finitely many stream derivatives. For each stream derivative $f_{\overline{w}}$, let the function $\widehat{f}_{\overline{w}}: (\mathbf{V}^{x+m})^\omega \rightarrow (\mathbf{V}^x)^\omega$ to be $\tau\sigma \mapsto \pi_0(f_{\overline{w}}(\tau\sigma))$. Note that each of these functions are causal and monotone, because they are constructed from pieces that are causal and monotone.

In particular, $\mu_f(\sigma)$ is the least fixed point of $\widehat{f}_\varepsilon((-)\sigma)$. Using the Kleene fixed point theorem, the least fixed point of $\widehat{f}_\varepsilon((-)\sigma)$ can be obtained by composing $\widehat{f}_\varepsilon((-)\sigma)$ repeatedly with itself. This means that $\mu_f(\sigma) = \bigsqcup_{k \in \mathbb{N}} \widehat{f}_\varepsilon^k(\perp^\omega, \sigma)$ where $\widehat{f}_\varepsilon^k$ is the k -fold composition of $f(-, \sigma)$ with itself, i.e. $\widehat{f}_\varepsilon^0(\tau\sigma) = \tau$

and $\widehat{f^{k+1}}(\tau\sigma) = \widehat{f}(\widehat{f^k}(\sigma, \tau))\sigma$. That the mapping μ_f is causal and monotone is straightforward: each of the functions in the join is causal and monotone, and join preserves these properties. It remains to show this mapping has finitely many stream derivatives.

When equipped with \preceq , the set of functions $(\mathbf{V}^{x+m})^\omega \rightarrow (\mathbf{V}^x)^\omega$ is a poset, of which $\{\widehat{f_w} \mid w \in (\mathbf{V}^{x+m})^*\}$ is a finite subset. Restricting the ordering \preceq to this set yields a finite poset. Since this poset is finite, the set of strictly increasing sequences in this poset is also finite. We will now demonstrate a relationship between these sequences and stream derivatives of μ_f .

Suppose $S = \widehat{f_{w_0}} \prec \widehat{f_{w_1}} \prec \cdots \prec \widehat{f_{w_{\ell-1}}}$ is a strictly increasing sequence of length ℓ in the set of stream functions $\{\widehat{f_w} \mid w \in (\mathbf{V}^{x+m})^*\}$. We define a function $g_S : (\mathbf{V}^m)^\omega \rightarrow (\mathbf{V}^x)^\omega$ as $(\sigma) \mapsto \bigsqcup_{k \in \mathbb{N}} g_k(\sigma)$ where

$$g_k(\sigma) = \begin{cases} \perp^\omega & \text{if } k = 0 \\ \widehat{f_{w_k}}((g_{k-1}(\sigma))\sigma) & \text{if } 1 \leq k \leq \ell. \\ \widehat{f_{w_{\ell-1}}}((g_{k-1}(\sigma))\sigma) & \text{if } \ell < k \end{cases}$$

Let the set $G := \{g_S \mid S \text{ is a strictly increasing sequence}\}$. When S is set to the one-item sequence \widehat{f} , g_S is μ_f , so $\mu_f \in G$. As G is finite, this means that if G is closed under stream derivative, μ_f has finitely many stream derivatives. Any element of G is either \perp^ω or has the form $\sigma \mapsto \widehat{f_{w_k}}(g_{k-1}(\sigma)\sigma)$ for some $\sigma \in (\mathbf{V}^m)^\omega$ and $k > 0$. As \perp^ω is its own stream derivative, we need to show that applying stream derivative to the latter produces another element of G .

$$\begin{aligned} \sigma \mapsto \left(\widehat{f_{w_k}}((g_{k-1}(\sigma))\sigma) \right)_{ab} &= \sigma \mapsto \text{tl} \left(\widehat{f_{w_k}}(ab :: (g_{k-1}(\sigma))\sigma) \right) \\ &= \sigma \mapsto \text{tl} (\pi_0 (\widehat{f_{w_k}}(ab :: (g_{k-1}(\sigma))\sigma))) \\ &= \sigma \mapsto \pi_0 (\text{tl} (\widehat{f_{w_k}}(ab :: (g_{k-1}(\sigma))\sigma))) \\ &= \sigma \mapsto \pi_0 ((\widehat{f_{w_k}}((g_{k-1}(\sigma))\sigma))_{ab}) \\ &= \sigma \mapsto \pi_0 (\widehat{f_{ab::w_k}}((g_{k-1}(\sigma))\sigma)) \\ &= \sigma \mapsto \widehat{f_{ab::w_k}}((g_{k-1}(\sigma))\sigma) \end{aligned}$$

As $\pi_0 (\widehat{f_{ab::w_k}})$ is in G , the latter is closed under stream derivative. Subsequently, μ_f has finitely many stream derivatives.

This means that all the components of $\sigma \mapsto \pi_1 (f(\mu_f(\sigma)\sigma))$ are causal, monotone and finitely specified, and as these properties are preserved by composition, the composite must also have them, so $\sigma \mapsto \pi_1 (f(\mu_f(\sigma)\sigma))$ is in $\mathbf{Stream}_{\mathcal{I}}$. To show that this is a trace, it remains to check the axioms of STMCs, which can all be shown to hold fairly easily. \square

Semantics can now be assigned to circuits in \mathbf{SCirc}_{Σ} with a traced PROP morphism into $\mathbf{Stream}_{\mathcal{I}}$.

Definition 26. For each $v \in \mathbf{V}$, let $\text{val}_v : (\mathbf{V}^0)^\omega \rightarrow \mathbf{V}^\omega$ be defined as $(\text{val}_v)[()] := v$ and $(\text{val}_v)_\emptyset := \text{val}_\perp$. Let $\text{shift}_v : \mathbf{V}^\omega \rightarrow \mathbf{V}^\omega$ be defined as $(\text{shift}_v)[\sigma] := v$ and $(\text{shift}_v)_{a::\sigma} := \text{shift}_a$.

Lemma 27. The stream functions in Def. 26 are causal, monotone and finitely specified.

Proof. For the first four functions, the k th element of the output stream is computed by a monotone operation on k th element of the input stream, so the stream function is monotone and causal. Since the k th input element cannot affect a later output element, there is one stream derivative: the original function. The function val_v has no inputs so it is trivially causal and monotone. It has one stream derivative: the stream function that constantly outputs \perp . The function shift_\perp is causal as the $i + 1$ th output element depends on only the i th element. There are $|\mathbf{V}|$ stream derivatives, as there is a different one for each possible input value. Each of these stream derivatives is monotone, as the initial output is fixed regardless of input, and, on input a , the stream derivative is the stream function that initially outputs a . \square

These stream functions are therefore morphisms in $\mathbf{Stream}_{\mathcal{I}}$, so are suitable candidates for the semantics of \mathbf{SCirc}_{Σ} .

Definition 28. Let $\llbracket - \rrbracket_{\mathcal{I}}^{\mathbf{S}} : \mathbf{SCirc}_{\Sigma} \rightarrow \mathbf{Stream}_{\mathcal{I}}$ be the traced PROP morphism defined as

$$\begin{aligned} \llbracket \boxed{-F} \rrbracket_{\mathcal{I}}^{\mathbf{S}}(\sigma)(k) &:= \llbracket \boxed{-F} \rrbracket_{\mathcal{I}}^{\mathbf{C}}(\sigma)(k) \\ \llbracket \boxed{v} \rrbracket_{\mathcal{I}}^{\mathbf{S}} &:= \text{val}_v \quad \llbracket \boxed{\square} \rrbracket_{\mathcal{I}}^{\mathbf{S}} := \text{shift}_\perp \end{aligned}$$

Given a sequential circuit $\boxed{-F}$, we say that the stream function $\llbracket \boxed{-F} \rrbracket_{\mathcal{I}}^{\mathbf{S}}$ is its behaviour under \mathcal{I} .

3.1 Monotone Mealy machines

Every circuit in \mathbf{SCirc}_Σ now has a denotation in the form of a stream function in $\mathbf{Stream}_\mathcal{T}$, computed using $\llbracket - \rrbracket_\mathcal{T}^S$. For completeness of the denotational semantics, we need to show that every stream function f in $\mathbf{Stream}_\mathcal{T}$ has a circuit in \mathbf{SCirc}_Σ with f as its denotation.

As a tool to help answer this question, we lift the well-known formalism of *Mealy machines* [Mea55] to lattices to yield *monotone Mealy machines*. Mealy machines are a common way of specifying the behaviour of sequential circuits [KJ09], and there is a homomorphism from every Mealy machine to a causal, finitely specified stream function [Rut06]. We establish that in the case of *monotone Mealy machines* this homomorphism is to *monotone stream functions*, establishing monotone Mealy machines as a bridge between circuits and stream functions.

Definition 29 (Mealy machine [Mea55]). *Let M and N be finite sets. A (finite) (M, N) -Mealy machine is a tuple (S, f, s_0) where S is a finite set called the state space, $f: S \rightarrow (S \times N)^M$ is the Mealy function, and $s_0 \in S$ is the start state.*

The sets M and N are the *inputs* and *outputs* of the machine. Given a state $s \in S$ and input $a \in M$, the Mealy function f produces a pair $f(s)(a) = \langle s', n \rangle$. We will use the shorthand $f_0 := (s, a) \mapsto \pi_0(f(s)(a))$ and $f_1 := (s, a) \mapsto \pi_1(f(s)(a))$ for the transition and output component of the Mealy function respectively.

Mealy machines can be viewed *coalgebraically*. A coalgebra of an endofunctor F is a pair of an object X and a morphism $X \rightarrow FX$. The first two components of a (M, N) -Mealy machine (S, f) is a coalgebra of the \mathbf{Set} -endofunctor $S \mapsto (S \times N)^M$. We call this a *Mealy coalgebra*; essentially the same as a Mealy machine but without a designated start state.

Example 30. In [BRS08], the notation $f(s)(a) = \langle s[a], s_a \rangle$ is also used to describe the Mealy function. Let Γ be the set of causal stream functions $M^\omega \rightarrow N^\omega$ for sets M and N , and let $\nu: \Gamma \rightarrow (\Gamma \times N)^M$ be the function defined as $\nu: (f, a) \mapsto \langle f_a, f[a] \rangle$. Then (Γ, ν) is an (M, N) -Mealy coalgebra.

A homomorphism h between two Mealy coalgebras (S, f) and (T, g) with interface (M, N) is a function $h: S \rightarrow T$ preserving transitions and outputs, i.e. for a Mealy function f a homomorphism h satisfies $h(f_0(s, a)) = f_0(h(s), a)$ and $h(f_1(s, a)) = f_1(s, a)$. The *final* Mealy coalgebra has a unique homomorphism from every other Mealy coalgebra: the coalgebra of streams defined in Ex. 30.

Proposition 31 ([Rut06], Prop. 2.2). *For every Mealy coalgebra (S, f) , there exists a unique homomorphism $!: (S, f) \rightarrow (\Gamma, \nu)$.*

Proof. A homomorphism $g: (S, f) \rightarrow (\Gamma, \nu)$ is a function $S \rightarrow \Gamma$, so for a state $s \in S$, $!(s)$ will be a stream function $M^\omega \rightarrow N^\omega$. For stream $\sigma \in M^\omega$, the elements of the stream $!(s)(\sigma)$ are the outputs f would produce given those inputs, starting from state s . \square

Not all Mealy machines (S, f, s) correspond to a digital circuit in \mathbf{SCirc}_Σ ; those that *do* can be identified by checking whether $!(s)$ lands in $\mathbf{Stream}_\mathcal{T}$. By definition, all stream functions in the image of $!(-)$ are causal, and since we only reason with *finite* Mealy machines we can also conclude the following:

Lemma 32. *Given a Mealy machine (S, f, s) , $!(s)$ is finitely specified.*

Proof. S is finite, and $!(-)$ must preserve transitions. \square

Monotonicity is more subtle, as the states of a Mealy machine are not naturally ordered. However, since each state corresponds to a stream function, they can inherit the ordering from Def. 22, and subsequently a notion of monotonicity.

Definition 33. *Let M and N be posets and let $f, g: M^\omega \rightarrow N^\omega$ be stream functions. We say $f \preceq g$ if $f(\sigma) \leq_{N^\omega} g(\sigma)$ for all $\sigma \in M^\omega$.*

Definition 34 (State order). *Given posets M, N and a (M, N) -Mealy machine (S, f, s_0) , we say that, for states $s, s' \in S$, $s \preceq s'$ if $!(s) \preceq !(s')$, where $!$ is the unique function $S \rightarrow \Gamma$.*

Definition 35 (Monotone Mealy machine). *Given posets (M, \leq_M) and (N, \leq_N) , an (M, N) -Mealy machine (S, f, s) is called *monotone* if f is monotone with respect to the appropriate orders.*

Lemma 36. *For a causal stream function $f: M^\omega \rightarrow N^\omega$, $a \mapsto f[a]$ and $a \mapsto f_a$ are monotone if and only if f is monotone.*

Proof. First the (\Leftarrow) direction. Recall that monotonicity implies that, for a given stream function g and inputs σ, τ , if $\sigma \leq \tau$ i.e. if $\sigma(i) \leq \tau(i)$ for all $i \in \mathbb{N}$, then $g(\sigma) \leq g(\tau)$ i.e. $g(\sigma)(i) \leq g(\tau)(i)$. Observe that this means that $g(a :: \sigma)(i) \leq g(b :: \sigma)(i)$ if $a \leq b$. Using this fact it is a simple exercise to show that the $a \mapsto g[a]$ and $a \mapsto g_a$ are monotone. Let $a, b \in M$ such that $a \leq b$. First we show $a \mapsto g[a]$ is monotone:

$$\begin{aligned} g[a] &= \text{hd}(g(a :: \sigma)) \\ &= g(a :: \sigma)(0) \\ &\leq g(b :: \sigma)(0) && \text{by monotonicity of } g \\ &= \text{hd}(g(b :: \sigma)) \\ &= g[b] \end{aligned}$$

And now we show that $a \mapsto g_a$ is monotone:

$$\begin{aligned} g_a(\sigma)(i) &= \text{tl}(g(a :: \sigma))(i) \\ &= g(a :: \sigma)(i+1) \\ &\leq g(b :: \sigma)(i+1) && \text{by monotonicity of } g \\ &= \text{tl}(g(b :: \sigma))(i) \\ &= g_b(\sigma)(i) \end{aligned}$$

Now the (\Rightarrow) direction. For a stream function g , assume that $a \mapsto g[a]$ and $a \mapsto g_a$ are monotone. We need to show that g is monotone, i.e. for streams σ, τ , if $\sigma \leq \tau$ then $g(\sigma)(i) \leq g(\tau)(i)$. Let $\sigma := w :: a :: \sigma'$ and $\tau := w' :: b :: \tau'$ where w is a finite sequence of length k . Since $\sigma \leq \tau$, $w \leq w'$ and $a \leq b$. Let g_w be the result of repeatedly obtaining the stream derivative for each element of w : this is also monotone by function composition. Therefore:

$$\begin{aligned} g(\sigma) &= g(w :: a :: \sigma')(k) \\ &= \text{tl}^k(g(w :: a :: \sigma'))(0) \\ &= g_w(a :: \sigma')(0) \\ &= g_w[a] \\ &\leq g_w[b] && \text{by monotonicity of } a \mapsto g[a] \\ &= g_w(b :: \tau)(0) \\ &\leq g_{w'}(b :: \tau)(0) && \text{by monotonicity of } a \mapsto g_a \\ &= \text{tl}^k(g(w' :: b :: \tau'))(0) \\ &= g(w' :: \tau')(k) \\ &= g(\tau) \end{aligned}$$

□

Lemma 37. For a monotone Mealy machine $S, f, s, !s$ is monotone.

Proof. The initial output and stream derivative of $!(S, f)(s)$ is defined as $a \mapsto s[a]$ and $a \mapsto s_a$ respectively: as these are monotone by definition of a monotone Mealy machine, the stream function is monotone by Lem. 36. □

In order to map between circuits, stream functions, and monotone Mealy machines we assemble the latter into a traced PROP. We use a standard notion of Mealy machine composition.

Definition 38 (Cascade product [Gin14]). Given an (M, N) -Mealy machine (S, f, s_0) and an (N, P) -Mealy machine (T, g, t_0) , their cascade product is a (M, P) -Mealy machine defined as

$$(S \times T, ((s, t), a) \mapsto ((f_0(s, a), g_0(t, f_1(s, a)), g_1(t, f_1(s, a))), (s_0, t_0)).$$

Definition 39. Let $\text{Mealy}_{\mathcal{I}}$ be the PROP with morphisms $m \rightarrow n$ the monotone $(\mathbf{V}^m, \mathbf{V}^n)$ -Mealy machines. Composition is by cascade product and tensor is by direct product.

Definition 40. Let (S, g, s_0) be a monotone $(\mathbf{V}^{x+m}, \mathbf{V}^{x+n})$ -Mealy machine and let $\mu_{s,a}$ be the least fixpoint of $r \mapsto \pi_0(g(s, ra))$ for fixed state $s \in S$ and input $a \in \mathbf{V}^m$. Then the least fixed point of (S, g, s_0) is a monotone $(\mathbf{V}^m, \mathbf{V}^n)$ -Mealy machine defined as $(S, (s, a) \mapsto g(s, (\mu_{s,a} a), s_0)$.

Proposition 41. *Def. 40 is a trace on $\mathbf{Mealy}_{\mathcal{I}}$.*

Proof. Since Mealy machines in $\mathbf{Mealy}_{\mathcal{I}}$ are monotone, their Mealy functions are also monotone, so they have a least fixed point. The axioms of STMCs can be shown to hold with this construction. \square

Example 42. Consider the monotone $(\mathbf{V}^3, \mathbf{V}^3)$ -Mealy machine with state set \mathbf{V}_B , initial state \perp , and Mealy function

$$g := (s, (v, u, w)) \mapsto \langle \neg(u \wedge v), \neg(s \wedge w), \neg(u \wedge v), \neg(s \wedge w) \rangle.$$

To take the trace of this machine, we must first compute the least fixed point of $v \mapsto \neg(s \wedge w)$, which is clearly just $\neg(s \wedge w)$. Therefore the Mealy function of the traced $(\mathbf{V}^2, \mathbf{V}^2)$ machine is $(s, (u, w)) \mapsto g(s, (\neg(s \wedge w), u, w))$.

As $\mathbf{Mealy}_{\mathcal{I}}$ is a STMC, we can define a traced PROP morphism to it from \mathbf{SCirc}_{Σ} .

Definition 43. Let $[-]_{\mathcal{I}}: \mathbf{SCirc}_{\Sigma} \rightarrow \mathbf{Mealy}_{\mathcal{I}}$ be the traced PROP morphism with action

$$\begin{aligned} [-\boxed{F}]_{\mathcal{I}} &:= \left(\{()\}, \bar{v} \mapsto \left((), \llbracket -\boxed{F} \rrbracket_{\mathcal{I}}^{\mathbf{C}}(\bar{v}) \right), () \right) \\ [\boxed{v}]_{\mathcal{I}} &:= (\{s_v, s_{\perp}\}, \{s_v \mapsto (s_{\perp}, v), s_{\perp} \mapsto (s_{\perp}, \perp)\}, s_v) \\ [-\boxed{v}]_{\mathcal{I}} &:= (\{s_v \mid v \in \mathbf{V}\}, (s_v, a) \mapsto (v, s_a), s_{\perp}) \end{aligned}$$

Example 44. Applying $[-]_{\mathcal{I}}$ to the SR NOR latch from Ex. 10 produces the monotone Mealy machine in Ex. 42.

It is essential that the translation between circuits and monotone Mealy machines *preserves behaviour*: if we translate a circuit $-\boxed{F}-$ into a monotone Mealy machine and then into a stream function, this stream function should be the behaviour of $-\boxed{F}-$.

Corollary 45. For a monotone Mealy machine $(S, f, s_0) \in \mathbf{Mealy}_{\mathcal{I}}, !(s_0) \in \mathbf{Stream}_{\mathcal{I}}$.

Definition 46. Let $!_{\mathcal{I}}(-): \mathbf{Mealy}_{\mathcal{I}} \rightarrow \mathbf{Stream}_{\mathcal{I}}$ be a PROP morphism sending a monotone Mealy machine (S, f, s_0) to $!(s_0)$, where $!$ is the unique homomorphism $(S, f) \rightarrow (\Gamma, \nu)$.

All that remains is to check that the translation from circuits to streams via monotone Mealy machines agrees with the direct translation from circuits to streams.

Theorem 47. $\llbracket - \rrbracket_{\mathcal{I}}^{\mathbf{S}} = !(-) \circ [-]_{\mathcal{I}}$.

Proof. Since morphisms of $\mathbf{Mealy}_{\mathcal{I}}$ and $\mathbf{Stream}_{\mathcal{I}}$ are both Mealy coalgebras, we just need to check that the transitions and outputs of the image of $[-]_{\mathcal{I}}$ and $\llbracket - \rrbracket_{\mathcal{I}}^{\mathbf{S}}$ agree. \square

3.2 Circuit synthesis

Using monotone Mealy machines, we will now define a map from a stream function f in $\mathbf{Stream}_{\mathcal{I}}$ into a circuit with f as its behaviour, thus showing that causal, monotone and finitely specified stream functions are a complete denotational semantics for sequential circuits. We do this by first recalling how to retrieve a (monotone) Mealy machine from a causal (monotone) stream function, then lift the technique for encoding Mealy machines as circuits [KJ09] to the *monotone* setting. Finally we will verify that this construction does indeed preserve behaviour.

As mentioned above, a causal stream function $f: M^{\omega} \rightarrow N^{\omega}$ is in fact a Mealy machine with interface (M, N) . Given such a function f , a minimal Mealy machine is obtainable.

Corollary 48 (Corollary 2.3, [Rut06]). For a causal, finitely specified stream function $f: M^{\omega} \rightarrow N^{\omega}$, let S be the least set of causal stream functions including f and closed under stream derivatives: i.e. for all $h \in S$ and $a \in M, h_a \in S$. Then the Mealy machine $\llbracket f \rrbracket_{\mathcal{I}} = (S, g, f)$, where $g(h)(a) = \langle h[a], h_a \rangle$, has the smallest state space of Mealy machines with the property $!_{\mathcal{I}}\llbracket f \rrbracket_{\mathcal{I}} = f$.

Proof. Since S is generated from the function f and is the *smallest* possible set, there are no unreachable states in S and no two states can ‘share the same behaviour’. \square

Lemma 49. Let $f: M^{\omega} \rightarrow N^{\omega}$ be a monotone causal stream function for posets M and N ; given the Mealy machine $\llbracket f \rrbracket_{\mathcal{I}} = (S, g, f)$ defined as in Cor. 48, the Mealy function g is monotone.

Proof. Lem. 36 shows these functions are monotone for fixed input letters: it remains to show that the functions are monotone for fixed functions from S . Let $h \in S$ and suppose $a \leq_M a'$. Since h is monotone, $h[a] = \text{hd}(h(a :: \sigma)) \leq_N \text{hd}(h(a' :: \sigma)) = h[a']$, and similarly for the transition function. As these functions are monotone in both components, they are monotone overall. \square

Corollary 50. *The map $\llbracket - \rrbracket_{\mathcal{I}}$ defined in Cor. 48 is a PROP morphism $\mathbf{Stream}_{\mathcal{I}} \rightarrow \mathbf{Mealy}_{\mathcal{I}}$.*

For regular Mealy machines, there is a standard procedure in circuit design [KJ09] in which each state of the Mealy machine is *encoded* as a power of values and combinational logic used to transform inputs into appropriate outputs.

For *monotone* Mealy machines, this procedure must respect monotonicity as the combinational logic is constructed using monotone components; an arbitrary encoding cannot be used. We will now show how to select a suitable encoding; recall from Def. 34 that the states in a monotone Mealy machine inherit an ordering from their corresponding stream functions.

Definition 51 (Monotone encoding). *Let S be a set with a partial order \preceq and total order \leq such that the elements of S form a chain $s_0 \leq s_1 \leq \dots \leq s_{k-1}$. The \leq -encoding for this assignment is a function $\gamma_{\leq}: S \rightarrow \mathbf{V}^k$ defined as $\gamma_{\leq}(s)(i) := \top$ if $s_i \preceq s$ and $\gamma_{\leq}(s)(i) := \perp$ otherwise.*

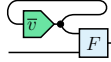
Example 52. Recall the monotone Mealy machine from Ex. 42, which has state set $\mathbf{V}_B := \{\perp, \text{t}, \text{f}, \top\}$. We choose the total order \leq on \mathbf{V}_B as $\perp \leq \text{t} \leq \text{f} \leq \top$; subsequently, the \leq -encoding is defined as $\perp \mapsto \top\perp\perp\perp, \text{t} \mapsto \top\top\perp\perp, \text{f} \mapsto \top\perp\top\perp, \top \mapsto \top\top\top\top$.

Lemma 53. *For an ordered state space (S, \preceq) and a \leq -encoding γ_{\leq} , $s \preceq s'$ if and only if $\gamma_{\leq}(s) \sqsubseteq \gamma_{\leq}(s')$.*

Proof. First the (\Rightarrow) direction. Let $s_i \preceq s_j$; we need to show that for every $l < k$, $s_i(l) \sqsubseteq s_j(l)$. The only way this can be violated is if $s_i(l) = \top$ and $s_j(l) = \perp$. But since $s_i \preceq s_j$, if $s_l \preceq s_i$ then $s_l \preceq s_j$ also holds.

Now the (\Leftarrow) direction. Assume that $\gamma_{\leq}(s_i) \sqsubseteq \gamma_{\leq}(s_j)$; we need to show that $s_i \preceq s_j$; i.e. that $\gamma_{\leq}(s_j)(i) = \top$ if $\gamma_{\leq}(s_i)(i) = \top$. If $\gamma_{\leq}(s_i)(i) = \top$, then for each $l < k$ then $\gamma_{\leq}(s_i)(l) \sqsubseteq \gamma_{\leq}(s_j)(l)$; in particular $\gamma_{\leq}(s_i)(i) \sqsubseteq \gamma_{\leq}(s_j)(i)$. By definition of γ_{\leq} , $\gamma_{\leq}(s_i)(i) = \top$, so if $\gamma_{\leq}(s_i) \sqsubseteq \gamma_{\leq}(s_j)$ then $\gamma_{\leq}(s_j)(i)$ is also \top . \square

The goal is to construct a combinational circuit morphism that, when interpreted as a function, implements the output and transition function of the Mealy machine. However, such a morphism may not exist for all interpretations.

Definition 54 (Functional completeness). *An interpretation \mathcal{I} of Σ is functionally complete if there exists a map $\llbracket - \rrbracket: \mathbf{Func}_{\mathcal{I}} \rightarrow \mathbf{SCirc}_{\Sigma}$ sending $f: \mathbf{V}^m \rightarrow \mathbf{V}^n$ to a circuit of the form  for some word $\bar{v} \in \mathbf{V}^*$ such that $\llbracket \llbracket f \rrbracket \rrbracket_{\mathcal{I}}^{\mathbf{S}}(\sigma)(i) = f(\sigma(i))$.*

For a given interpretation there may be many such maps, but we will assume there is a fixed procedure $\llbracket - \rrbracket$ and refer to a circuit $\llbracket f \rrbracket$ as the *normalised circuit* for f .

Remark 55. Note that $\llbracket - \rrbracket$ maps to \mathbf{SCirc}_{Σ} , as the function $\llbracket f \rrbracket$ could require the use of values, a sequential component.

Example 56. The Belnap interpretation from Ex. 14 is functionally complete; see Appendix A.1 for the details.

To retrieve a circuit from a monotone Mealy machine in a functionally complete interpretation, we use the Mealy function.

Definition 57 (Monotone completion). *For lattices M, N, P such that $M \subseteq N$, and a monotone function $f: M \rightarrow P$, let the monotone completion of f be the function $f_m: N \rightarrow P$ recursively defined as*

$$f_m(v) = \begin{cases} f(v) & \text{if } v \in M \\ \perp & \text{if } v = \perp^m, \perp \notin M \\ \bigsqcup \{f_m(w) \mid w \leq_N v\} & \text{otherwise} \end{cases}$$

Definition 58 (Monotone Mealy encoding). *For a monotone Mealy machine (S, f, s_0) with k states and a monotone encoding γ_{\leq} , a monotone Mealy encoding is a function $\gamma_{\leq}(f): \mathbf{V}^k \times \mathbf{V}^m \rightarrow \mathbf{V}^k \times \mathbf{V}^n$ defined as the monotone completion of the function $(\gamma_{\leq}(s), \bar{x}) \mapsto (\gamma_{\leq}(f_0(s, \bar{x})), f_1(s, \bar{x}))$.*

Lemma 59. *A monotone Mealy encoding is in $\mathbf{Func}_{\mathcal{I}}$.*

Now the latter:

$$\begin{aligned}
\phi \circ \psi &= !(-) \circ [-]_{\mathcal{I}} \circ ||-||_{\mathcal{I}}^{\leq} \circ \langle\langle - \rangle\rangle_{\mathcal{I}} \\
&= [[-]]_{\mathcal{I}}^{\mathbf{S}} \circ ||-||_{\mathcal{I}}^{\leq} \circ \langle\langle - \rangle\rangle_{\mathcal{I}} && \text{Thm. 47} \\
&= !(-) \circ \langle\langle - \rangle\rangle_{\mathcal{I}} && \text{Thm. 64} \\
&= \text{id}_{\mathbf{Stream}_{\mathcal{I}}} && \text{Cor. 48}
\end{aligned}$$

□

There is no isomorphism between \mathbf{SCirc}_{Σ} and $\mathbf{Stream}_{\mathcal{I}}$ as many circuits may have the same semantics but different syntax.

Definition 66 (Denotational equivalence). *We say that two sequential circuits are denotationally equivalent under \mathcal{I} , written $m\text{-}\boxed{F}\text{-}^n \approx_{\mathcal{I}} m\text{-}\boxed{G}\text{-}^n$ if $[[\text{-}]]_{\mathcal{I}}^{\mathbf{S}} = [[\text{-}]]_{\mathcal{I}}^{\mathbf{S}}$. Let $\mathbf{SCirc}_{\Sigma/\approx_{\mathcal{I}}}$ be the result of quotienting \mathbf{SCirc}_{Σ} by $\approx_{\mathcal{I}}$.*

Corollary 67. *For functionally complete interpretations, $\mathbf{SCirc}_{\Sigma/\approx_{\mathcal{I}}} \cong \mathbf{Stream}_{\mathcal{I}}$.*

This confirms that, for a functionally complete interpretation \mathcal{I} , the PROP $\mathbf{Stream}_{\mathcal{I}}$ of causal, finitely specified and monotone stream functions, is a *sound and complete* semantic domain for sequential circuits: every circuit in \mathbf{SCirc}_{Σ} has a stream function in $\mathbf{Stream}_{\mathcal{I}}$ and every function $f \in \mathbf{Stream}_{\mathcal{I}}$ has a class of circuits with f as their behaviour.

4 Operational semantics

The behaviour of two circuits in \mathbf{SCirc}_{Σ} can be compared by examining their corresponding stream functions. However, translating a circuit into a stream function obscures the internal structure of the circuit, much like representing a combinational circuit by its truth table. To better relate behaviour and structure in circuits we now define an *operational semantics* which evaluates circuits in a stepwise manner. An informal operational semantics was presented in [GJL17] but only for the case of *closed* circuits with *delay-guarded* feedback; in this section we drop these two requirements and present a sound and complete operational semantics for *all* sequential circuits.

An operational semantics is defined in terms of *reductions*. Here we are motivated by *mechanising* circuit reduction; for a set of reductions to be suitable then there should be a terminating strategy for evaluating inputs to circuits.

Notation 68 (Reduction). *A reduction from $m\text{-}\boxed{F}\text{-}^n$ to $m\text{-}\boxed{G}\text{-}^n$, is denoted $m\text{-}\boxed{F}\text{-}^n \rightsquigarrow m\text{-}\boxed{G}\text{-}^n$. If there are reductions $m\text{-}\boxed{F}\text{-}^n \rightsquigarrow m\text{-}\boxed{G}\text{-}^n \rightsquigarrow \dots \rightsquigarrow m\text{-}\boxed{H}\text{-}^n$, we write $m\text{-}\boxed{F}\text{-}^n \rightsquigarrow^* m\text{-}\boxed{H}\text{-}^n$. A reduction $m\text{-}\boxed{F}\text{-}^n \rightsquigarrow m\text{-}\boxed{G}\text{-}^n$ is sound if $[[\text{-}]]_{\mathcal{I}}^{\mathbf{S}} = [[\text{-}]]_{\mathcal{I}}^{\mathbf{S}}$.*

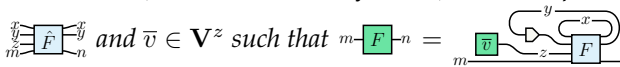
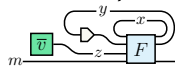
A reduction is effectively a directed equation, which we will apply to circuits in \mathbf{SCirc}_{Σ} . This means that the only equations that hold ‘on the nose’ are the axioms of STMCs; diagrams can be deformed in order to expose redexes. To mechanise the reduction process it is also preferable to appeal to the graphical notation and use *graph rewriting* [GK23].

4.1 Non-delay-guarded feedback

Often it is the case that one exhaustively applies some set of reductions until a normal form is reached. When reasoning with circuits, the presence of the trace means we need to be slightly more careful; for example, one could end up infinitely unfolding the trace.

We will instead present a more ‘guided’ operational semantics which can still be used to produce the outputs of a circuit. First we will define some global transformations used to bring a circuit into a suitable form for evaluation.

Lemma 69 (Global trace-delay form). *For a sequential circuit $m\text{-}\boxed{F}\text{-}^n$ there exists a combinational circuit*

 *and $\bar{v} \in \mathbf{V}^z$ such that $m\text{-}\boxed{F}\text{-}^n =$  by axioms of STMCs.*

Proof. By applying the axioms of traced categories. □

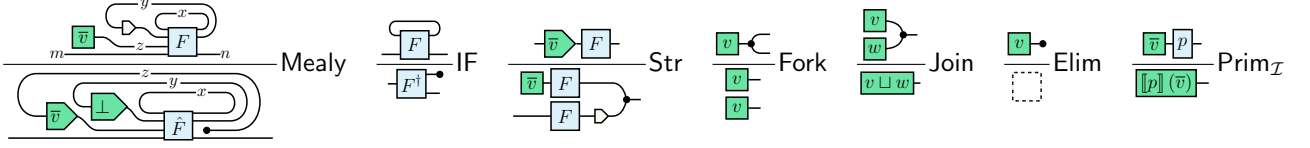


Figure 4: Productive reductions

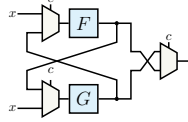


Figure 5: A cyclic combinational circuit with useful output [MSB12, Fig. 1]

This form is evocative of what we saw when mapping from Mealy machines to circuits in the previous section, but state is defined using delays and values rather than registers.

Definition 70 (Pre-Mealy form). *A sequential circuit is in Mealy form if it is in the form $m \text{---} \boxed{F} \text{---} n$.*

Lemma 71. *The Mealy rule in Fig. 4 is sound.*

Proof. It is a simple exercise to check the corresponding stream functions. \square

Corollary 72. *For any sequential circuit $m \text{---} \boxed{F} \text{---} n$, there exists at least one valid application of the Mealy rule.*

There is still the non-delay-guarded trace to consider. Circuits that do *not* have this trace are of particular interest.

Definition 73 (Mealy form). *A sequential circuit $m \text{---} \boxed{F} \text{---} n$ is in Mealy form if it is in the form $m \text{---} \boxed{S} \text{---} \boxed{F} \text{---} n$.*

We still do not have all the pieces required to translate all circuits into Mealy form.

Example 74. Consider the circuit $\boxed{t} \text{---} \boxed{D} \text{---}$ using the signature Σ_B from Ex. 2. In the stream semantics, this evaluates to $(\perp \wedge t) \sqcup ((\perp \wedge t) \wedge t) \sqcup \dots = \perp$. In the syntactic realm, the instant feedback blocks us from reaching Mealy form.

Remark 75. In circuit design, it is common to enforce that circuits have no non-delay-guarded feedback; one might ask should we too should enforce this in order to stick to ‘well-behaved’ circuits. Not only would this violate the categorical setting of a STMC (the ‘yanking’ equation would no longer hold), careful use of non-delay-guarded feedback can still result in useful circuits as a clever way of sharing resources [Mal94; Rie04; MSB12]. The minimal circuit to implement a function often *must* be constructed using cycles [Riv77; RB03]!

Example 76. Consider the circuit in Fig. 5 where \boxed{F} and \boxed{G} are arbitrary combinational circuits. The trapezoidal gate is a *multiplexer*, which has a vertical *control* input and horizontal *data* inputs. When the control is f the output is the first data input and when it is t the output is the second data input. (The behaviour when the control is \perp or \top is not important here.) The interpretation of the multiplexer is

$$\begin{aligned} \left[\boxed{f} \right] (f, x, y) &= x & \left[\boxed{\perp} \right] (\perp, x, y) &= \perp \wedge (x \vee y) \\ \left[\boxed{t} \right] (t, x, y) &= y & \left[\boxed{\top} \right] (\top, x, y) &= \top \wedge (x \vee y) \end{aligned}$$

This circuit is already in pre-Mealy form, and has non-delay-guarded feedback. Despite this, it produces useful output when the control signal is 0 or 1:

$$\begin{aligned} \left[\boxed{F} \right]_{\mathcal{I}_*}^S (\sigma)(i) &= \left[\boxed{F} \boxed{G} \right]_{\mathcal{I}_*}^C (\pi_2(\sigma(i))) \text{ if } \pi_0(\sigma(i)) = 0 \\ \left[\boxed{F} \right]_{\mathcal{I}_*}^S (\sigma)(i) &= \left[\boxed{G} \boxed{F} \right]_{\mathcal{I}_*}^C (\pi_2(\sigma(i))) \text{ if } \pi_0(\sigma(i)) = 1 \end{aligned}$$

A circuit with no delay-guarded feedback implements a (combinational) function as it has no delays. We need a way of eliminating ‘instant feedback’ from combinational circuits; using a methodology also applied in [RB12], we turn to the Kleene fixed-point theorem.

Lemma 77. For a monotone function $f: \mathbf{V}^{n+m} \rightarrow \mathbf{V}^n$ and $i \in \mathbb{N}$, let $f^i: \mathbf{V}^m \rightarrow \mathbf{V}^n$ be defined as $f^0(x) = f(\perp^n, x)$ and $f^{k+1}(x) = f(f^k(x), x)$. Let c be the length of the longest chain in the lattice \mathbf{V}^n . Then, for $j > c$, $f^c(x) = f^j(x)$.

Proof. Since f is monotone, it has a least fixed point by the Kleene fixed-point theorem. This will either be some value v or, since \mathbf{V} is finite, the \top element. The most iterations of f it would take to obtain this fixpoint is c , i.e. the function produces a value one step up the lattice each time. \square

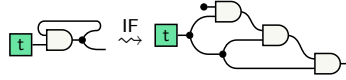
Definition 78 (Iteration). For a combinational circuit \mathbb{F}_m^n , let its n th iteration \mathbb{F}_m^n be defined inductively over n as $\mathbb{F}_m^0 := \mathbb{F}_m$ and $\mathbb{F}_m^{k+1} := m \rightarrow \mathbb{F}_m^k \rightarrow \mathbb{F}_m^n$.

Definition 79 (Unrolling). For an interpretation with values \mathbf{V} , the unrolling of a combinational circuit \mathbb{F}_m^n , written \mathbb{F}_m^c , is defined as \mathbb{F}_m^{c+1} where c is the length of the longest chain in \mathbf{V}^x .

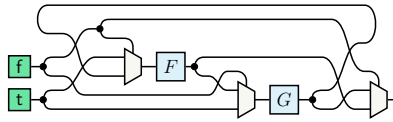
Proposition 80. The instant feedback rule IF in Fig. 4 is sound.

Proof. By Lem. 77, applying the circuit c times reaches a fixpoint. The circuit is combinational so each element of the output $\llbracket \mathbb{F} \rrbracket_{\mathcal{I}}^S(\sigma)(i)$ is a function: Lem. 77 can be applied to each element. \square

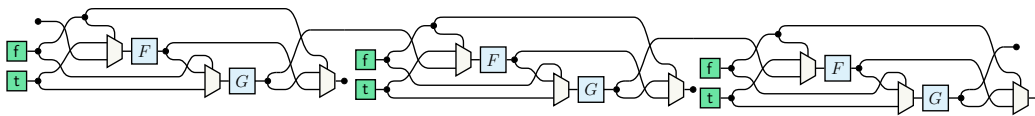
Example 81. Recall Ex. 74; applying the IF rule unrolls the trace.



Example 82. Recall the cyclic combinational circuit from Ex. 76. When applied to some inputs, this can also be reduced appropriately using IF. We will precompose the circuit with values so it only produces interesting output on the first tick, and then reduce it using equations in \mathcal{C} .



Note that the control switch is set to f . We then apply IF to eliminate the feedback loop.



This is a circuit with the same semantics as the original circuit, but cycle-free.

Any circuit can now be brought into Mealy form.

Theorem 83. For a sequential circuit \mathbb{F}_m^n , there exist at least one combinational circuit \mathbb{F}_m^c and values $s \in \mathbf{V}^x$ such that $\mathbb{F}_m^n \rightsquigarrow^* m \rightarrow \mathbb{F}_m^c \rightarrow \mathbb{F}_m^n$ by applying Mealy followed by IF.

Proof. In $\text{SCirc}_{\Sigma/\mathcal{E}_{\mathcal{I}}}$, any sequential circuit is equal to a circuit in pre-Mealy form by Def. 70. Then, since the core is a combinational circuit with a non-delay-guarded trace, it is equal to a circuit without a non-delay-guarded trace by IF. \square

If applied locally for every feedback loop, the IF equation would cause an exponential blowup. However, if a circuit is in global trace-delay form, the equation need only be applied once to the global loop; although the value of c increases as the number of feedback wires increases, it only does so linearly in the height of the lattice.

Example 84. Fig. 6 applies Mealy and IF to the the SR latch circuit from Ex. 10, as the longest chain in \mathbf{V}_B is 2.

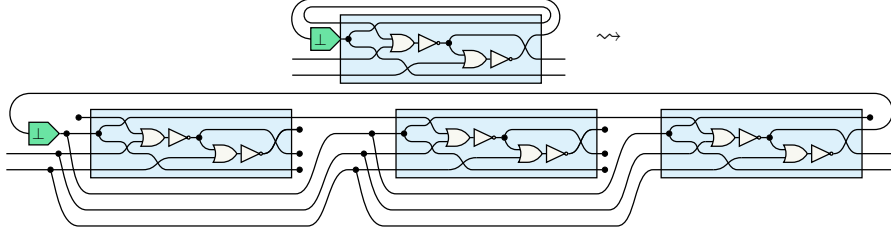


Figure 6: Applying Mealy and IF to the SR latch circuit

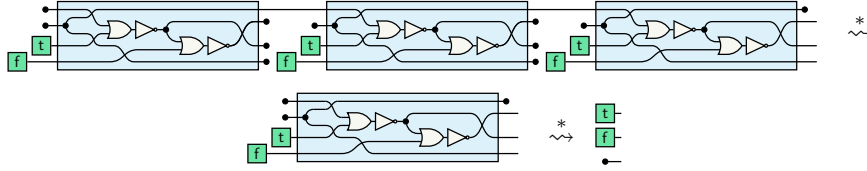


Figure 7: Applying combinational reductions to the ‘now’ copy of the circuit in Fig. 6 with inputs tf .

4.2 Productivity

Given a circuit, one common task is to feed it some inputs and *simulate* its outputs, so the behaviour of the circuit can be verified. This corresponds to finding reductions such that $m-\bar{v}\rightarrow F^{-n} \rightsquigarrow^* m-G\rightarrow \bar{w}^{-n}$ for any sequential circuit and input values. We will first consider the combinational case.

Lemma 85. *The streaming rule Str in Fig. 4 is sound.*

The streaming rule says that when a combinational circuit has an input with an instant and a delayed component, the circuit can be copied so that one copy handles what is happening ‘now’ and the other handles what is happening ‘later’. The final rules will reduce the ‘now’ copy to values.

Lemma 86 (Value rules). *The value rules Fork, Join, Elim and Prim $_{\mathcal{I}}$ listed in Fig. 4 are sound.*

This is the only step in which exhaustive application is required, as more is involved than just copying circuit components.

Lemma 87. *Applying the productive rules is confluent.*

Proof. There are no overlaps between the rules. □

Lemma 88. *For a combinational circuit \boxed{F} and $\bar{v} \in \mathbf{V}^m$, there exists $\bar{w} \in \mathbf{V}^n$ such that applying the productive value rules exhaustively to $\bar{v}\boxed{F}$ terminates at \bar{w} .*

The streaming and the value rules enable a circuit in Mealy form to process inputs.

Corollary 89. *For circuit $m-\bar{v}\rightarrow \boxed{F}^{-n}$ there exist $\bar{t} \in \mathbf{V}^x$ and $\bar{w} \in \mathbf{V}^n$ such that $m-\bar{v}\rightarrow \boxed{F}^{-n} \rightsquigarrow^* m-\bar{t}\rightarrow \boxed{F}^{-n} \rightarrow \bar{w}^{-n}$ by applying Str once followed by the value rules exhaustively.*

Example 90. Fig. 7 shows how the ‘now’ copy of the transformed SR latch circuit from Fig. 6 for inputs tf (a ‘reset’ pulse) is reduced by the combinational rules. The next state is t , the first output is f and the second is \perp . The first output (false) is what we would expect given a reset pulse, but the second may raise an eyebrow. This arises due to the delay; recall that this models inertial delay in the wires rather than an actual memory element. Subsequently, it will take another cycle to produce the expected output ft .

By combining the rules of the previous section with the strategy in this one, we have a procedure for processing inputs to a circuit.

Corollary 91 (Productivity). *For sequential circuit $m-\boxed{F}^{-n}$ and inputs $\bar{v} \in \mathbf{V}^m$, there exists $\bar{w} \in \mathbf{V}^n$ such that $m-\bar{v}\rightarrow \boxed{F}^{-n} \rightsquigarrow^* m-G\rightarrow \bar{w}^{-n}$ by applying Mealy, IF and Str once successively followed by the value rules exhaustively.*

4.3 Observational equivalence

Earlier we presented the notion of *denotational equivalence*, in which circuits are equivalent if their stream semantics are equal. Another way of showing equivalence is that of *observational equivalence* [Mor69]. Observational (or *extensional*) equivalence is the notion that two processes are equivalent if they cannot be distinguished solely by their input-output behaviour.

Testing equivalence is traditionally shown by checking that a program behaves the same in all *contexts*.

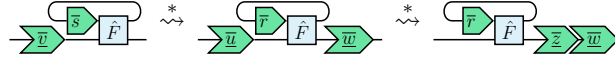
Notation 92 (Waveform). The empty waveform is defined as $n \xrightarrow{\varepsilon} n := n \square n$. Given values $\bar{v} \in \mathbf{V}^n$ and sequence $\underline{\bar{w}} \in (\mathbf{V}^n)^*$, the waveform for sequence $\bar{v} :: \underline{\bar{w}}$ is drawn as $n \xrightarrow{\bar{v} :: \underline{\bar{w}}} n := \xrightarrow{\bar{v}} \xrightarrow{\underline{\bar{w}}}$.

Observational equivalence also has a relational definition [Gor98]. A relation is *adequate* if it only relates circuits that have the same denotational semantics; observational equivalence is the largest adequate congruence relation. To define such a relation for digital circuits, we will consider the inputs needed to ‘fully evaluate’ the behaviour of a circuit.

Lemma 93. Let $\square F$ be a sequential circuit with c delay components. Then applying Cor. 91 successively to a Mealy form of this circuit will produce at most $|\mathbf{V}|^c$ unique states.

Proof. The only varying elements of the state word are contributed by the c delay components, as the values transition to \perp . \square

Corollary 94. Given a circuit in Mealy form $\square F$ and input sequence $\underline{\bar{v}} \in (\mathbf{V}^m)^*$ of length $|\mathbf{V}|^c + 1$, there exists a state $\bar{r} \in \mathbf{V}^x$, an input sequence $\underline{\bar{u}} \in (\mathbf{V}^m)^*$ and output sequences $\underline{\bar{w}}, \underline{\bar{z}} \in (\mathbf{V}^n)^*$ such that applying Cor. 91 yields the following reduction pattern:



Proof. By Lem. 93. \square

With this in mind, we can now define an adequate notion of observational equivalence for sequential circuits.

Definition 95 (Observational equivalence of circuits). We say that two sequential circuits $m \square F$ and $m \square G$ with no more than c delays are said to be *observationally equivalent* under \mathcal{I} , written $\square F \sim_{\mathcal{I}} \square G$ if applying productivity produces the same output waveforms for all input waveforms $\underline{\bar{v}} \in (\mathbf{V}^m)^*$ of length $|\mathbf{V}|^c + 1$.

Theorem 96. Given two sequential circuits $m \square F$ and $m \square G$, $m \square F \sim_{\mathcal{I}} m \square G$ if and only if $\llbracket m \square F \rrbracket_{\mathcal{I}}^{\mathbf{S}} = \llbracket m \square G \rrbracket_{\mathcal{I}}^{\mathbf{S}}$.

Proof. The (\Rightarrow) direction follows by Cor. 94, as every possible internal configuration of the circuit will be tested. For (\Leftarrow) , if $\llbracket m \square F \rrbracket_{\mathcal{I}}^{\mathbf{S}} = \llbracket m \square G \rrbracket_{\mathcal{I}}^{\mathbf{S}}$, then this means $\llbracket m \square F \rrbracket_{\mathcal{I}}^{\mathbf{S}}(\underline{\bar{v}} :: \sigma) = \llbracket m \square G \rrbracket_{\mathcal{I}}^{\mathbf{S}}(\underline{\bar{v}} :: \sigma)$ for any $\sigma, \tau \in (\mathbf{V}^m)^{\omega}$. By definition of $\llbracket - \rrbracket_{\mathcal{I}}^{\mathbf{S}}$, we then have that $\llbracket m \xrightarrow{\underline{\bar{v}}} \square F \rrbracket_{\mathcal{I}}^{\mathbf{S}}(\sigma) = \llbracket m \xrightarrow{\underline{\bar{v}}} \square G \rrbracket_{\mathcal{I}}^{\mathbf{S}}(\sigma)$. Since this holds for *all* sequences $\underline{\bar{v}}$, it must hold for those of length $|\mathbf{V}|^c + 1$, so the condition for observational equivalence is met. \square

Corollary 97. $\sim_{\mathcal{I}}$ is the largest adequate congruence on \mathbf{SCirc}_{Σ} .

Proof. For $\sim_{\mathcal{I}}$ to be a congruence it must be preserved by composition, tensor and trace, and for it to be the largest there must be no denotationally equal circuit it does not relate. These both follow by Thm. 96. \square

This makes $\sim_{\mathcal{I}}$ a suitable notion of observational equivalence for sequential circuits.

Definition 98. Let $\mathbf{SCirc}_{\Sigma/\sim_{\mathcal{I}}}$ be defined as $\mathbf{SCirc}_{\Sigma}/\sim_{\mathcal{I}}$.

Corollary 99. There is an isomorphism $\mathbf{SCirc}_{\Sigma/\approx_{\mathcal{I}}} \cong \mathbf{SCirc}_{\Sigma/\sim_{\mathcal{I}}}$.

$$\curvearrowright = \text{---} \text{ (M1)} \quad \curvearrowleft = \text{---} \text{ (M2)} \quad \bullet \dashv \text{---} = \bullet \text{---} \text{ (BD)} \quad \boxed{F} = \boxed{F^\dagger} \text{---} \text{ (IF)}$$

Figure 8: Set of Mealy equations \mathcal{M} .

5 Algebraic semantics

The previous section gives an upper bound on the length of waveforms required to establish observational equivalence, so we have a terminating strategy for comparing the behaviour of digital circuits using a pseudo-normal form. Unfortunately, this is still an *exponential* upper bound, so it is infeasible to check for equivalence of circuits with more than a few delay components.

It is often the case that circuits differ by only a few components; perhaps one is trying to show that two similar implementations are the same. In this case, it is more feasible to find the parts of the circuit that differ and then check if they have the same behaviour. With the syntactic representation, we can reason about these subcircuits *algebraically* using equations. The final contribution of this paper is to define a *sound and complete* algebraic semantics for sequential circuits.

Remark 100. An ‘equational theory’ was presented in [GJ16], but the equations were used to quotient the syntax as an ad-hoc semantics, and soundness and completeness were not considered. Here we use the stream semantics to guide our choice in equations.

5.1 Mealy equations

We will begin by reframing the Mealy rules as equations for translating circuits into Mealy form. Rather than porting the large Mealy rule across directly, we will express it in terms of smaller equations.

Definition 101. Let \mathcal{M} be the set of equations in Fig. 8.

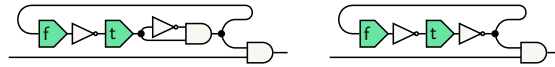
Proposition 102. Given a sequential circuit \boxed{F} , there exists a circuit in Mealy form such that $\boxed{F} = \boxed{F^\dagger}$ in $\mathbf{SCirc}_\Sigma / \mathcal{M}$.

Proof. Registers are created from delays by $\dashv \text{---} \stackrel{\text{M1}}{=} \curvearrowright$ and values by $\boxed{v} \stackrel{\text{M2}}{=} \curvearrowleft \stackrel{\text{BD}}{=} \bullet \dashv \text{---} \boxed{v}$. \square

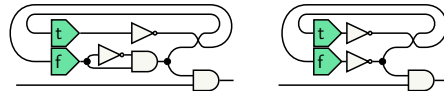
5.2 Normalising circuits

One might wonder if completeness of an equational theory could be established by translating two circuits to Mealy forms with the same state, and then using combinational equations to translate between the combinational cores. Unfortunately, this is not sufficient.

Example 103. Consider the following circuit in $\mathbf{SCirc}_{\Sigma_B}$:



The registers will *always* contain f and t and both circuits will produce a constant f output, so a complete equational theory should be able to translate between them. To this end, we assemble these into Mealy forms with the same initial states:

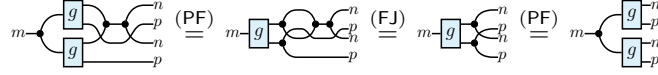


The combinational cores do *not* have the same semantics! They only act the same because they receive certain inputs from \mathbf{V}^3 .

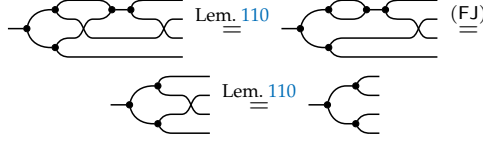
This example shows that we need to take into the *context* when defining equations; equations that only deal with the interactions between individual generators will not suffice.

Instead, we will define a family of equations for translating between *bisimilar* circuits. Recall that in a functionally complete interpretation, there is a *normalised* circuit $\|f\| \in \mathbf{SCirc}_\Sigma$ for any function f . With the right set of equations, the combinational core of a circuit in Mealy form can be translated into such a canonical form; from this one can read off a truth table.

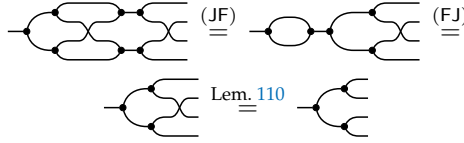
Proof. This is by induction over the structure of the circuit \boxed{F} . First we must check the base cases. For \boxed{g} :



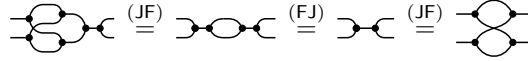
For $\boxed{\text{C}}$ we first check if only one of the outputs are joined:



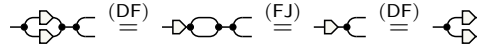
And now if both outputs are joined:



For $\boxed{\text{D}}$:



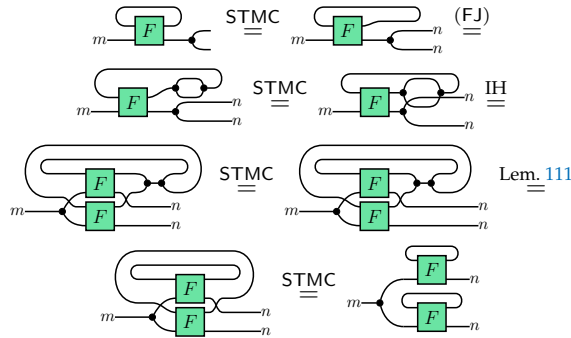
For $\boxed{\text{E}}$:



The proofs for $\boxed{\text{A}}$, $\boxed{\text{B}}$ and $\boxed{\text{X}}$ are trivial, as are the inductive cases for composition and tensor. For the inductive case for the trace, axioms of STMCs are applied in reverse to create two ‘global traces’, and then the inductive hypothesis is applied to reach the final result. \square

Proposition 112. $\text{SCirc}_{\Sigma}/\mathcal{C} + \text{IF}$ is Cartesian.

Proof. We need to show that the two equations in Fig. 9 hold. The naturality of the copy for the generators, composition and tensor is immediate by the relevant equations in Fig. 10. For trace it is more involved:



The naturality of the stub for the generators, composition and tensor is again immediate by equations in Fig. 10. For the trace the circuit can be brought into global trace-delay form, followed by using (IF) and (DD). \square

Cartesian categories allow us to ‘isolate’ outputs.

Lemma 113. In a Cartesian category, $m\text{-}\boxed{F}\text{-}_n^x = \text{Circuit}$.

Proof. $m\text{-}\boxed{F}\text{-}_n \stackrel{(M1,M2)}{=} \text{Circuit} \stackrel{(NC)}{=} \text{Circuit}$ \square

By repeatedly applying Lem. 113 to a circuit in Mealy form, a copy of the core \boxed{F} can be made for the transition and output.

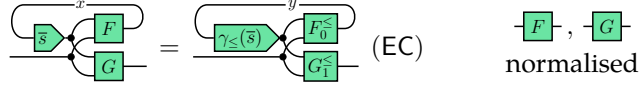


Figure 11: The encoding equation

Lemma 114. *In $\mathbf{SCirc}_{\Sigma}/(\mathcal{M} + \text{IF} + \mathcal{C} + \mathcal{N}_{\mathcal{I}})$, any sequential circuit $m\text{-}\boxed{F}^n$ is equal to a circuit in normalised Mealy form.*

Proof. Any circuit can be brought to Mealy form using $\mathcal{M} + \text{IF}$ by Thm. 83; normalised Mealy form follows using $\mathcal{C} + \mathcal{N}_{\mathcal{I}}$. \square

5.3 Circuit encodings

The normalised Mealy form may not be unique: even if the states are the same size, there are multiple orderings.

In the construction of a circuit from a Mealy machine using $\|\cdot\|_{\mathcal{I}}^{\leq}$, each state was encoded as a word containing only \perp and \top values. The equations we will now add translate any circuit in normalised Mealy form using such an encoding. First we compute the possible state words of a circuit.

Definition 115 (Circuit states). *Given a circuit in normalised Mealy form $m\text{-}\boxed{F}^n$, let $S(F_0, \bar{s}) \subseteq \mathbf{V}^x$ be the smallest set containing \bar{s} and closed under $\bar{r} \mapsto F_0^{-1}(\bar{r}, \bar{v})$ for any $\bar{v} \in \mathbf{V}^m$.*

For encodings as in Def. 51, a circuit with y states must be translated into a circuit with a state word in \mathbf{V}^y .

Definition 116 (Translations). *Let $S \subseteq \mathbf{V}^x$ be a set of states with total order \leq , and let $\gamma_{\leq} : S \rightarrow \mathbf{V}^y$ be the induced \leq -encoding.*

For a circuit $m\text{-}\boxed{F}^n := \|f\|$, its γ_{\leq} -transition is the circuit $m\text{-}\boxed{F_0^{\leq}}^n := \|(\bar{s}, \bar{v}) \mapsto \gamma_{\leq}(f(\gamma_{\leq}^{-1}(\bar{s}), \bar{v}))\|$.

Similarly, for a circuit $m\text{-}\boxed{G}^n := \|g\|$, its γ_{\leq} -output is the circuit $m\text{-}\boxed{F_1^{\leq}}^n := \|(\bar{s}, \bar{v}) \mapsto g(\gamma_{\leq}^{-1}(\bar{s}), \bar{v})\|$.

Proposition 117 (Encoding equation). *For two normalised circuits $m\text{-}\boxed{F}^x$ and $m\text{-}\boxed{G}^n$ along with $\bar{s} \in \mathbf{V}^x$, let \leq be an ordering on $S(F, \bar{s})$. Then the equation in Fig. 11 is sound.*

Proof. The interpretations of $m\text{-}\boxed{F_0^{\leq}}^x$ and $m\text{-}\boxed{G_1^{\leq}}^n$ are as the behaviours of $m\text{-}\boxed{F}^x$ and $m\text{-}\boxed{G}^n$ wrapped in appropriate decodings and encodings. \square

The encoding equation transforms a normalised Mealy form into one where the state is just \perp and \top values.

Definition 118. *Let $\mathcal{E}_{\mathcal{I}}$ be defined as $\mathcal{M} + \text{IF} + \mathcal{C} + \mathcal{N}_{\mathcal{I}} + \text{EC}$, and let $\mathbf{SCirc}_{\Sigma/\mathcal{E}_{\mathcal{I}}}$ be defined as $\mathbf{SCirc}_{\Sigma}/\mathcal{E}_{\mathcal{I}}$.*

With this set of equations, we have a sound and complete algebraic semantics for sequential circuits.

Theorem 119. *In a functionally complete interpretation \mathcal{I} , $m\text{-}\boxed{F}^n = m\text{-}\boxed{G}^n$ in $\mathbf{SCirc}_{\Sigma/\mathcal{E}_{\mathcal{I}}}$ if and only if $\llbracket \boxed{F} \rrbracket_{\mathcal{I}}^{\mathbf{S}} = \llbracket \boxed{G} \rrbracket_{\mathcal{I}}^{\mathbf{S}}$.*

Proof. All the equations are sound, so we only need to consider the (\Leftarrow) direction. By Lem. 114, the two circuits can be brought to normalised Mealy form using $\mathcal{M} + \text{IF} + \mathcal{C} + \mathcal{N}_{\mathcal{I}}$. By Def. 62 and Cor. 65 there

must exist a circuit $m\text{-}\boxed{H}^n := \| \begin{array}{c} \boxed{T} \\ \boxed{O} \end{array} \|$ such that $\llbracket \boxed{F} \rrbracket_{\mathcal{I}}^{\mathbf{S}} = \llbracket \boxed{H} \rrbracket_{\mathcal{I}}^{\mathbf{S}} = \llbracket \boxed{G} \rrbracket_{\mathcal{I}}^{\mathbf{S}}$. The circuit $m\text{-}\boxed{H}^n$ is encoded such that the state words are in the image of γ_{\leq} . This means that applying the encoding equation with \leq to the normalised Mealy forms obtained above will yield the circuit $m\text{-}\boxed{H}^n$. \square

Corollary 120. $\mathbf{SCirc}_{\Sigma/\approx_{\mathcal{I}}} \cong \mathbf{SCirc}_{\Sigma/\mathcal{E}_{\mathcal{I}}}$.

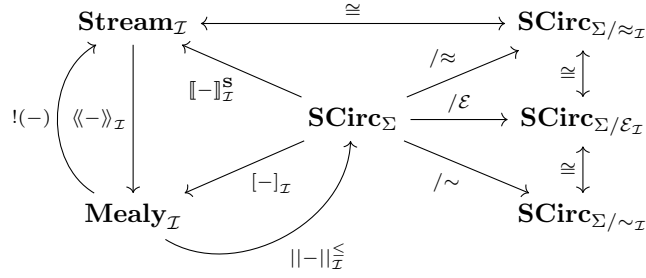




Figure 12: Relating categorical models of circuits

6 Conclusion

We have presented three ways of showing equivalence of circuits: *denotationally* by using the stream semantics, *operationally* by using the productive reduction strategy, and *algebraically* by using the sound and complete equational theory. This completes the research programme initiated by [GJ16; GJL17]. A summary of the categories involved is shown in Fig. 12.

String diagrams as a graphical syntax for monoidal categories were introduced a few decades ago [JS91; JSV96], and there has since been an explosion in their use for various applications, such as cyclic lambda calculi [Has97], fixpoint operators [Has03], quantum protocols [AC04], signal flow diagrams [BSZ14; BSZ15], linear algebra [BSZ17; Zan15; Bon+19; BP22], dynamical systems [BE15; FSR16], electrical circuits [BS22] and automatic differentiation [Alv+23]. While all these frameworks use compositionality in some way, the nature of digital circuits mean they differ to ours. In many of the above applications, the join and the fork form a *Frobenius structure*, making the wires bidirectional. This means the trace is constructed as , which degenerates to  in our *Cartesian* setting. Indeed, in any compact closed category the product would automatically be a coproduct (biproduct), which is a degeneracy incompatible with models of digital circuits.

There are other settings that permit loops but retain unidirectionality of wires. *Categories with feedback* were introduced in [KSW02] as a weakening of STMCs that removes the yanking axiom, enforcing that *all* traces are delay-guarded. In [Di+21] Mealy machines are characterised as a category with feedback: this is compatible with our framework since all ‘instant feedback’ is expressed as fixpoints and only delay-guarded feedback remains. *Categories with delayed trace* [SK19] weaken the notion further by removing the sliding axiom; this prohibits the unfolding rule so would be unsuitable.

Axiomatising fixpoint operators has been studied extensively [BÉ93; Ste00; SP00]. Since any Cartesian traced category admits a fixpoint (or *Conway*) operator [Has97], these equations can be expressed using the Cartesian equations and axioms of STMCs. Since our work takes place in a *finite* lattice, we are able express a fixpoint by iterating the circuit a finite number of times. While this result is well-known from the denotational perspective [SLG94], it has not been used before, perhaps surprisingly, to solve the problem of combinational feedback. The interplay of causal streams and dataflow categories has also been studied elsewhere: recently, a generalisation of causal streams known as *monoidal streams* [DS22] has been developed to provide semantics to dataflow programming. Although this generalises some aspects of this paper, our approach differs in the use of the finite lattice and monotone functions.

The correspondence between Mealy machines and digital circuits is a fundamental result in automata theory [Mea55] applied extensively in circuit design [KJ09]. The links between Mealy machines and causal stream functions using coalgebras is a more recent development [Rut05a; Rut05b; Rut06]. Mealy machines over meet-semilattices are introduced in [BRS08] to model a logical framework which includes a fixpoint. We also employ this technique in order to handle fixpoints, but also assemble (monotone) Mealy machines into a PROP in order to bridge between stream functions and sequential circuits.

String diagrams are not efficient to work with computationally. Instead they must be translated into combinatorial graphs; this was touched on in [GJL17] using *framed point graphs* [Kis12]. Recent work in string diagram rewriting [Bon+22a; Bon+22b; Bon+22c] has used *hypergraphs* for rewriting modulo Frobenius structure. This framework has been adapted for categories with a (co)monoid structure [FL23; MPZ23], and *traced* comonoid structure [GK23]: the setting in which we model digital circuits. We have already begun to develop an automated framework for rewriting circuits based on this work.

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A Results for Belnap logic

The interpretation used throughout this paper has been Belnap’s four value logic. In this section we will sketch out how this interpretation satisfies the requirements to be suitable for the fully compositional framework.

A.1 Functional completeness

The results of Section 3.2 and Section 5 depend on an interpretation being *functionally complete* (Def. 54).

Definition 121. Let $\mathbf{B} := \{0, 1\}$ be the set of Boolean values, and let $\wedge_{\mathbf{B}}, \vee_{\mathbf{B}}, \neg_{\mathbf{B}}$ be the usual operations on Booleans.

Lemma 122. The set $\{0, 1, \wedge_{\mathbf{B}}, \vee_{\mathbf{B}}, \neg_{\mathbf{B}}\}$ is functionally complete for all functions $\mathbf{B}^m \rightarrow \mathbf{B}$.

Proof. Let $f : \mathbf{B}^m \rightarrow \mathbf{B}$ be a Boolean function: we need to create a Boolean expression using variables v_0, v_1, \dots, v_{m-1} . For each $\bar{v} \in \mathbf{B}^m$, we construct a conjunction of all m variables, in which v_i is negated if $\bar{v}(i) = 0$. We can then define a disjunction of the conjunctions for words \bar{v} such that $f(\bar{v}) = 1$. If there are no such words, then the expression is 0. It is simple to check that this expression is equivalent to the original function. \square

We now turn our attention to Belnap values. The aim is to show completeness of the Belnap functions using the known results for Boolean functions. First, we shed some light on what a Belnap value really is.

Lemma 123. There is an isomorphism $\mathbf{V} \cong \mathbf{B}^2$.

Proof. There are several mappings one could choose, but for the purpose of this section we will use $\phi := \perp \mapsto 00, \text{f} \mapsto 10, \text{t} \mapsto 01, \top \mapsto 11$. \square

The Belnap values f and \top are *falsy*; they contain false information. Similarly, the Belnap values t and \top are *truthy*: they contain true information. The value \perp is neither falsy nor truthy. This is reflected in the mapping shown above; $\phi(v)(0)$ is 1 if and only if v is falsy, and $\phi(v)(1)$ is 1 if and only if v is truthy. We write $\phi_0(v) := \phi(v)(0)$ and $\phi_1(v) := \phi(v)(1)$.

This illuminates a new path: rather than trying to divine an expression directly from a Belnap function, we can instead define *two* functions; one for how falsy the output is, and one for how truthy it is.

Definition 124. Let $\mathbf{V}_0 := \{\perp, f\}$ and let $\mathbf{V}_1 := \{\perp, t\}$.

Note that for $v \in \mathbf{V}_0$, $\phi(v)(1) = 0$, and for $v' \in \mathbf{V}_1$, $\phi(v')(0) = 0$.

Lemma 125. \mathbf{V}_0 and \mathbf{V}_1 are closed under \wedge and \vee .

Proof. This can be verified by inspecting the truth tables:

\wedge	\perp	f
\perp	\perp	f
f	f	f

\vee	\perp	f
\perp	\perp	\perp
f	\perp	f

\wedge	\perp	t
\perp	\perp	\perp
t	\perp	t

\vee	\perp	t
\perp	\perp	t
t	t	t

□

These should truth tables should look familiar; indeed, if one squints a little we recover the truth tables for $\vee_{\mathbf{B}}$ and $\wedge_{\mathbf{B}}$ on the first row, and for $\wedge_{\mathbf{B}}$ and $\vee_{\mathbf{B}}$ on the second! This means that any expression we make using $\wedge_{\mathbf{B}}$ and $\vee_{\mathbf{B}}$ in the Boolean realm can be ‘simulated’ in the two Belnap subsets. Formally, we have the following.

Lemma 126. The following diagrams commute:

$$\begin{array}{ccc}
 (\mathbf{V}_0)^2 & \xrightarrow{\wedge} & \mathbf{V}_0 \\
 (\phi_0, \phi_0) \downarrow & & \downarrow \phi \\
 \mathbf{B}^2 & \xrightarrow{\vee_{\mathbf{B}}} & \mathbf{B}
 \end{array}
 \qquad
 \begin{array}{ccc}
 (\mathbf{V}_0)^2 & \xrightarrow{\vee} & \mathbf{V}_0 \\
 (\phi_0, \phi_0) \downarrow & & \downarrow \phi \\
 \mathbf{B}^2 & \xrightarrow{\wedge_{\mathbf{B}}} & \mathbf{B}
 \end{array}$$

$$\begin{array}{ccc}
 (\mathbf{V}_1)^2 & \xrightarrow{\wedge} & \mathbf{V}_1 \\
 (\phi_1, \phi_1) \downarrow & & \downarrow \phi_0 \\
 \mathbf{B}^2 & \xrightarrow{\wedge_{\mathbf{B}}} & \mathbf{B}
 \end{array}
 \qquad
 \begin{array}{ccc}
 (\mathbf{V}_1)^2 & \xrightarrow{\vee} & \mathbf{V}_1 \\
 (\phi_1, \phi_1) \downarrow & & \downarrow \phi_1 \\
 \mathbf{B}^2 & \xrightarrow{\vee_{\mathbf{B}}} & \mathbf{B}
 \end{array}$$

Proof. By testing the four values in each case. □

The eager reader might now assume we can proceed as with Lem. 122, constructing a disjunctive normal form from truth tables, but we replace Boolean operations with the appropriate Belnap ones. The more cautious reader may have noticed we have not discussed how the Boolean $\neg_{\mathbf{B}}$ can be simulated using Belnap operations. This is because it is simply not possible to do this while remaining in the two Belnap subsets! Fortunately, there is a certain subset of Boolean functions that can be constructed *without* using $\neg_{\mathbf{B}}$.

Definition 127. Let the total order $\leq_{\mathbf{B}}$ be defined as $0 \leq 1$.

As with \mathbf{V} , \mathbf{B}^m inherits the order on \mathbf{B} pointwise. Subsequently, a Boolean function $f: \mathbf{B}^m \rightarrow \mathbf{B}$ is *monotone* if $f(v) \leq_{\mathbf{B}} f(w)$ whenever $\bar{v} \leq_{\mathbf{B}} \bar{w}$. Intuitively, flipping an input bit from 0 to 1 can never flip an output bit from 1 to 0.

Lemma 128. The set of Boolean operations $\{\wedge, \vee, 1\}$ is functionally complete for monotone functions $\mathbf{B}^m \rightarrow \mathbf{B}$.

Proof. This progresses as with Lem. 122, but if the element of a word $\bar{v}(i) = 0$, it is omitted from the conjunction rather than the variable being negated.

To show that this expresses the same truth table as the original function, consider an omitted variable v_i ; there exists an assignment of the other variables such that if $v_i = 0$ then $f(\dots, v_i, \dots) = 1$. By

monotonicity, it must be the case that if $v_i = 1$ then $f(\dots, v_i, \dots) = 1$, so no information is lost by omitting the negation.

If $f(0, 0, \dots, 0) = 1$, then the inner conjunction is empty and must instead be represented by the constant 1, (the unit of $\wedge_{\mathbf{B}}$). This is again valid due to monotonicity, as if f produces 1 for the infimum, then it must produce 1 for all inputs. \square

Corollary 129. *The set $\{\wedge, \vee, f\}$ is functionally complete for monotone functions $(\mathbf{V}_0)^m \rightarrow \mathbf{V}_0$, and the set $\{\wedge, \vee, t\}$ is functionally complete for monotone functions $(\mathbf{V}_1)^m \rightarrow \mathbf{V}_1$.*

Proof. As there is an order isomorphism $\mathbf{V}_0 \cong \mathbf{V}_1 \cong \mathbf{B}$, any monotone function in the Belnap subsets can be viewed as a monotone Boolean function. This means the strategy of Lem. 128 can be applied using Lem. 126 to substitute the appropriate Belnap operation. \square

All the pieces are now in place to express the final functional completeness result; we just need to ‘explode’ a Belnap value into its falsy and truthy components, and then unify the two at the end.

Definition 130. *Let the functions*

$$\psi_0^0, \psi_0^1: \mathbf{V} \rightarrow \mathbf{V}_0 \quad \psi_1^0, \psi_1^1: \mathbf{V} \rightarrow \mathbf{V}_1$$

be defined according to the tables below.

	ψ_0^0	ψ_0^1	ψ_1^0	ψ_1^1
\perp	\perp	\perp	\perp	\perp
t	\perp	f	\perp	t
f	f	\perp	t	\perp
\top	f	f	t	t

The functions ψ_0^0 and ψ_0^1 sends a Belnap value v to f or t respectively if v is falsy. Similarly, ψ_1^0 and ψ_1^1 sends a Belnap value v to f or t if v is truthy. Otherwise, they produce \perp .

Lemma 131. *The functions in Def. 130 can be constructed using the set $\{\wedge, \vee, \neg, \perp\}$.*

Proof. From left to right, the columns in the table above are the functions $v \mapsto - \wedge \perp$, $v \mapsto \neg(- \vee \perp)$, $v \mapsto \neg(- \wedge \perp)$ and $v \mapsto - \vee \perp$. \square

Definition 132. *Let $f: \mathbf{V}^m \rightarrow \mathbf{V}$ be a monotone function. Then, let $f_0: ((\mathbf{V}_0)^m)^2 \rightarrow \mathbf{V}_0$ be defined as $f_0(\psi_0^0(\bar{v}), \psi_0^1(\bar{v})) := \phi_0(f(\bar{v}))$. Similarly, let $f_1: ((\mathbf{V}_1)^m)^2 \rightarrow \mathbf{V}_1$ be defined as $f_1(\psi_1^0(\bar{v}), \psi_1^1(\bar{v})) := \phi_1(f(\bar{v}))$.*

Theorem 133. *The set $\{\wedge, \vee, \neg, \sqcup, \perp, t, f\}$ is functionally complete for monotone functions $\mathbf{V}^m \rightarrow \mathbf{V}$.*

Proof. This follows by defining a function with the same behaviour as the original, but made up of components known to be expressible using the operations specified.

Let $f': \mathbf{V}^m \rightarrow \mathbf{V}^2$ be defined as

$$f'(\bar{v}) := (f_0(\psi_0^0(\bar{v}), \psi_0^1(\bar{v})), f_1(\psi_1^0(\bar{v}), \psi_1^1(\bar{v}))).$$

By Cor. 129, f_0 and f_1 can be defined using $\{\wedge, \vee, t, f\}$, and by Lem. 131, $\psi_0^0, \psi_0^1, \psi_1^0$ and ψ_1^1 can be defined using $\{\wedge, \vee, \perp\}$.

The output of $f'(\bar{v})$ is $(\phi_0(f(\bar{v})), \phi_1(f(\bar{v})))$ by definition; the falsiness and the truthiness of $f(\bar{v})$. To combine the two outputs into a single output we want to implement the following truth table:

\perp	\perp	\perp
\perp	t	t
f	\perp	f
f	t	\top

But this is clearly just the truth table for \sqcup , so the entire expression can be defined using $\{\wedge, \vee, \neg, \sqcup, \perp, t, f\}$. \square

Example 134. Consider the following truth table (in fact just the table for \neg).

\perp	\perp
t	f
f	t
\top	\top

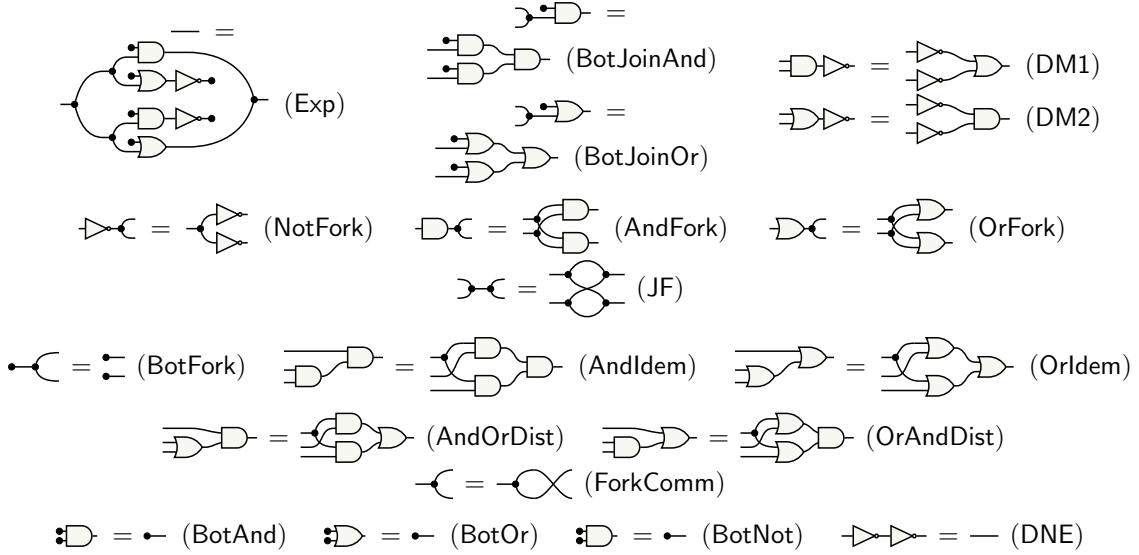


Figure 13: Set \mathcal{X} of explosion equations

We translate these into the falsy and truthy tables as follows:

$\perp\perp$	\perp	$\perp\perp$	\perp
$\perp f$	f	$\perp t$	\perp
$f\perp$	\perp	$t\perp$	t
ff	f	tt	t

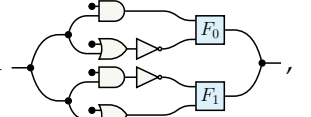
Using Cor. 129, the corresponding Belnap expressions are $(v_0, v_1) \mapsto v_0 \wedge (v_0 \vee v_1)$ and $(v_0, v_1) \mapsto v_1 \vee (v_0 \wedge v_1)$. Adding in the translations from Def. 130 and the join at the end, the overall Belnap expression becomes $(v_0, v_1) \mapsto (v_0 \wedge \perp) \wedge ((v_0 \wedge \perp) \vee \neg(v_1 \vee \perp)) \sqcup (v_1 \vee \perp) \vee (\neg(v_0 \wedge \perp) \wedge (v_1 \vee \perp))$.

Corollary 135. *The set of Belnap operations $\{\wedge, \vee, \neg, \perp, \sqcup\}$ is sufficient to implement all monotone functions $\mathbf{V}^{m+1} \rightarrow \mathbf{V}^n$.*

Proof. By repeating the process in Thm. 133 for each output. □

Since an expression can be defining using only operations with counterparts in the syntactic realm, it is

possible to define a map $\|\cdot\|_{\mathbb{B}} : \mathbf{Func}_{\mathcal{I}_{\mathbb{B}}} \rightarrow \mathbf{SCirc}_{\Sigma_{\mathbb{B}}}$ such that $\|f\|_{\mathbb{B}}$ is a term of the form



in which $\boxed{F_0}$ and $\boxed{F_1}$ are ‘syntactic’ conjunctive and disjunctive normal forms respectively. The exact definition of these terms is fiddly and tedious to define, but one can easily imagine what they might look like.

A.2 Canonical form

For a sound and complete equational theory, equations required to bring any combinational syntactic circuit into a form defined above.

Definition 136. *Let the set \mathcal{X} of explosion equations be defined as in Fig. 13*

Most of the equations in \mathcal{X} are well-known; the only interesting one is (Exp). This says that we can always ‘explode’ a wire into (trivial) falsy and truthy subcircuits before joining them back together. While on its own this may not seem very useful, when combined with the other equations it sets the stage for translating an entire circuit into an exploded form.

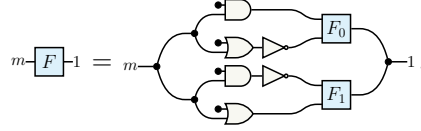
Lemma 137. *The equations in Fig. 13 are sound.*

Proof. By checking all the inputs. □

Lemma 138. For any combinational Belnap circuit $m\text{-}\boxed{F}\text{-}n$, the equation $\boxed{F}\text{-}\leftarrow = \begin{array}{c} \boxed{F} \\ \boxed{F} \end{array}$ in $\mathbf{CCirc}_{\Sigma_B}/\mathcal{X}$.

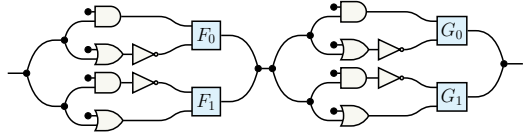
Proof. This follows for the base cases by applying AndFork, OrFork, NotFork and JF; the inductive cases are then trivial. \square

Proposition 139. Given a combinational Belnap circuit $m\text{-}\boxed{F}\text{-}1$, there exists combinational Belnap circuits $4m\text{-}\boxed{F_0}\text{-}1$ and $4m\text{-}\boxed{F_1}\text{-}1$ containing no $\boxed{\rightarrow}$ or $\boxed{\leftarrow}$ generators, such that



Proof. If \boxed{F} is just the identity, then it can be transformed into the desired form with (Exp). Since \boxed{F} has codomain 1 it cannot be a symmetry. For the other generators, (Exp) can first be applied to the output wire to create the exploded ‘skeleton’. The $\boxed{\rightarrow}$, $\boxed{\leftarrow}$, and $\boxed{\leftarrow}$ generators can then be pushed inside using the other equations in Fig. 13. A similar principle holds for $\boxed{\rightarrow}$, although as the negation ‘flips’ the translators using (DM1) and (DM2), (ForkComm) must be used to restore the correct order of gates, and (DNE) used to eliminate the additional $\boxed{\rightarrow}$ gates.

For composition, we assume that the two subcircuits are in the desired form, so we have



By Lem. 138, the first circuit can be propagated across the forks at the start of the second circuit, each of the four ‘translators’ has as input a copy of the first circuit. Using the same strategy as for the base case the components of the circuit can then be propagated across the translators. To complete the proof we need to ensure there is exactly one of each translator. Duplicates are handled by Lem. 138. Translators are flipped between using $\boxed{\rightarrow}$ and $\boxed{\leftarrow}$ when a $\boxed{\rightarrow}$ is propagated across them, but since the same circuit is applied to each translator the same flip will happen in reverse elsewhere. Therefore the eventual circuit will be in the correct form.

For tensor, the circuits can be interleaved using axioms of STMCs. \square

This form already looks very similar to a circuit in the image of $\|\text{-}\|$ in that it is the join of two circuits prefixed by ‘translators’. However, these circuits are not in conjunctive or disjunctive normal form. Fortunately, there are standard equations for translating ‘ordinary’ Boolean expressions (i.e. those without joins) into disjunctive and conjunctive normal forms.

Definition 140. Let \mathcal{F} be defined as the set of normal form equations listed in Fig. 14.

Definition 141. A circuit \boxed{F} is a conjunction if it only contains $\boxed{\rightarrow}$ gates and no symmetries, and a disjunction if it only contains $\boxed{\leftarrow}$ gates and no symmetries. A circuit \boxed{G} is in disjunctive normal form if it can be expressed as a tensor of conjunctions composed with a disjunction, and is in conjunctive normal form if it can be expressed as a tensor of disjunctions composed with a conjunction.

Proposition 142. Let \boxed{F} be a combinational Belnap circuit containing no $\boxed{\rightarrow}$ or $\boxed{\leftarrow}$ generators. Then there exists a circuit \boxed{G} containing only forks and symmetries and a Belnap circuit \boxed{H} in conjunctive normal form such that $\boxed{F} = \boxed{G}\text{-}\boxed{H}$ in $\mathbf{SCirc}_{\Sigma_B}/\mathcal{F}$. Similarly, there exists a circuit $\boxed{G'}$ containing only forks and symmetries and a circuit $\boxed{H'}$ in disjunctive normal form such that $\boxed{F} = \boxed{G'}\text{-}\boxed{H'}$ in $\mathbf{SCirc}_{\Sigma_B}/\mathcal{P}$.

Proof. The circuits \boxed{H} and $\boxed{H'}$ can be obtained using well-known procedures for obtaining disjunctive or conjunctive normal form; propagating the ‘root’ gate through the ‘leaf’ gates using distributivity, and using the other equations to ‘tidy up’ the term until it is in the desired normal form. The terms of forks and symmetries is then the result of propagating the other gates to the left. \square

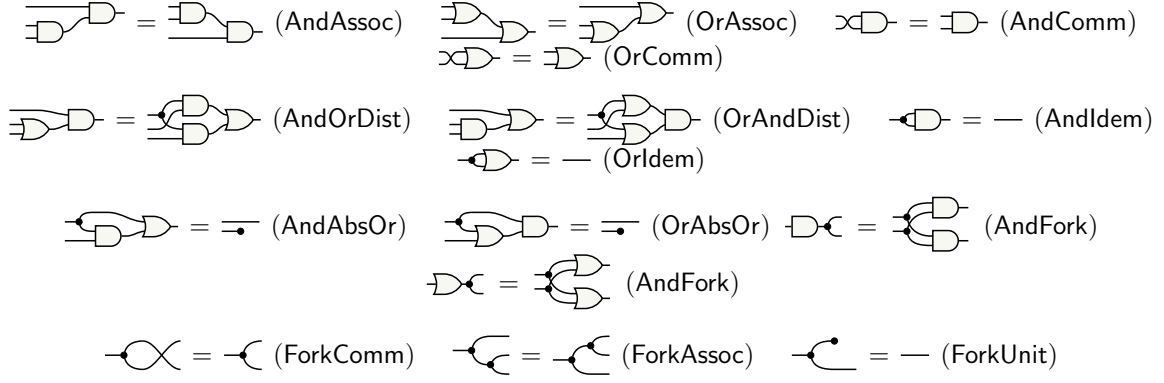


Figure 14: Set \mathcal{F} of normal form equations.

Putting these two results together gives us the desired canonical form theorem.

Theorem 143. Let $\|\cdot\|_{\mathbb{B}}: \mathbf{Func}_{\mathcal{I}_{\mathbb{B}}} \rightarrow \mathbf{CCirc}_{\Sigma_{\mathbb{B}}}$ be a map induced by functional completeness of the Belnap interpretation in the previous section. Given a combinational Belnap circuit $\lfloor F \rfloor$, there exists a circuit $\lfloor G \rfloor$ in the image of $\|\cdot\|_{\mathbb{B}}$ such that $\lfloor F \rfloor = \lfloor G \rfloor$ in $\mathbf{CCirc}_{\Sigma_{\mathbb{B}}}/\mathcal{X} + \mathcal{F}$.

Proof. Prop. 139 and Prop. 142 can be applied in sequence to create a circuit of the correct form. By using (ForkUnit, ForkAssoc and ForkComm) the initial construct of forks and symmetries can be adjusted until it is in the image of $\|\cdot\|_{\mathbb{B}}$, however it was initially defined. \square